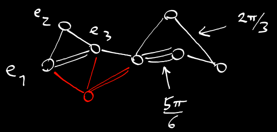
Classification.

- Si un diagramme est admissible, alors si on lui retire un sommet, il l'est encore.



(preuve en exercice)

- Prenons un diagramme adm. $\|e_i\| = 1$

$$2\cos(e_i,e_j) = \begin{cases} 0 & \text{non reliés} \\ -1 \\ -\sqrt{2} \\ -\sqrt{3} \end{cases}$$

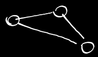
$$0 < \left\| \sum e_i \right\|^2 = \left(\sum e_i , \sum e_i \right)$$

$$= \sum (e_i,e_i) + 2 \sum_{i<j} (e_i,e_j)$$

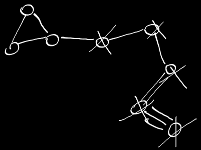
$$= n + \sum_{i<j} \underbrace{2\cos(e_i,e_j)}$$

$$1 \leqslant n - [\#\text{ paires des nœuds connectés}]$$

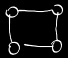
$$\boxed{[\#\text{ paires reliées}] \leqslant n-1}$$ → Les diagrammes adm sont des arbres

- 

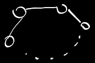
Non !



Non !

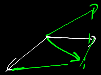


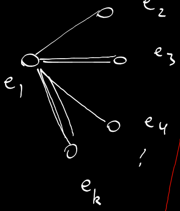
Non !



Non !

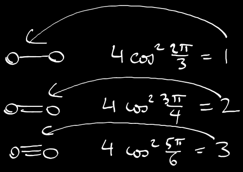
- Prenons e_1





famille orthonormée !

$$1 = \|e_1\|^2 > \sum_{i=2}^{k} (e_1,e_i)^2$$



$4\cos^2\frac{2\pi}{3} = 1$

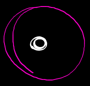
$4\cos^2\frac{3\pi}{4} = 2$

$4\cos^2\frac{5\pi}{6} = 3$

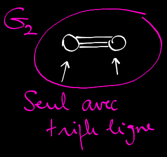
$$4 > \sum_{i=2}^{k} \left[4(e_1,e_i)^2\right] = [\#\text{ lignes}] \geqslant 0.$$

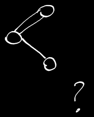
$\#$ lignes arrivant à un nœud est dans $\{\cancel{0}, 1, 2, 3\}$

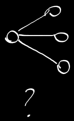
- Si c'est 0 :



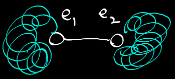
- Si c'est 3 :



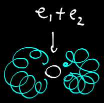




- Si

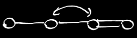


admissible alors

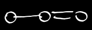


aussi.

(Ex :



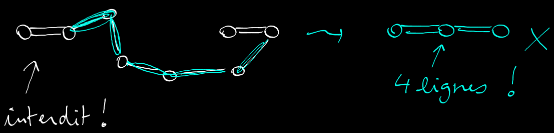
adm ⇒



aussi)

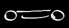
$$\|e_1 + e_2\|^2 = \|e_1\|^2 + \|e_2\|^2 + 2(e_1, e_2)$$
$$= 1 + 1 + 2\left(-\tfrac{1}{2}\right) = 1$$

Conséquences :



Il y a au plus 1 double ligne !

- Supposons qu'il y a une double ligne



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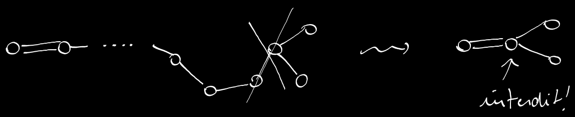
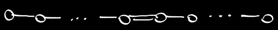
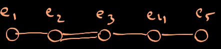


Seules possibilités :





est interdit !

$v = e_1 + 2e_2$ $\qquad \|v\|^2 = 1 + 4 + 2 \times 2 \times \left(-\tfrac{1}{2}\right) = 3$

$w = 3e_3 + 2e_4 + e_5$ $\qquad \|w\|^2 = 9 + 4 + 1 - 6 - 2 = 6$

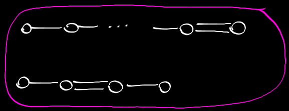
$$(v,w) = \left(e_1 + 2e_2\,,\ 3e_3 + 2e_4 + e_5\right)$$
$$= 2 \times 3 \times \left(-\tfrac{\sqrt{2}}{2}\right)$$
$$= -3\sqrt{2}.$$

$$(v,w)^2 < \|v\|^2 \cdot \|w\|^2$$

$18 < 3 \times 6$ ← cas d'égalité de C.S.

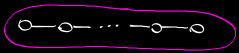
Faux

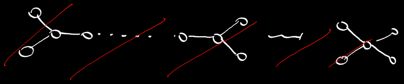
Conclusion : les seuls admissibles avec double ligne :



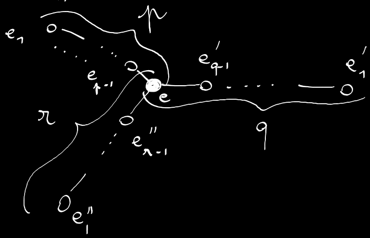
(et ils le sont !)

- Reste ceux avec uniquement des simples.

* Soit pas de bifurcation 

* Si au moins 2 bifurcations : 

* $\exists!$ bifurcation :



$$\begin{cases} u = e_1 + 2e_2 + \dots + (p-1)e_{p-1} \\ v = e'_1 + \dots + (q-1)\,e'_{q-1} \\ w = e''_1 + \dots + (r-1)\,e''_{r-1} \end{cases}$$

$$u' = \frac{u}{\|u\|}$$

$$v' = \frac{v}{\|v\|}$$

$$\|u\|^2 = \frac{p(p-1)}{2} \qquad \|v\|^2 = \frac{q(q-1)}{2} \qquad \|w\|^2 = \frac{r(r-1)}{2}$$

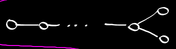
$$1 = \|e\|^2 > \underset{\displaystyle \downarrow \atop \displaystyle \frac{2(p-1)^2}{p(p-1)} \times \frac{1}{4}}{(e,u')^2} + (e,v')^2 + (e,w')^2$$

$$2 > \frac{p-1}{p} + \frac{q-1}{q} + \frac{r-1}{r} = 3 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)$$

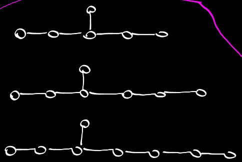
$\boxed{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1}$ $2 \leqslant p \leqslant q \leqslant r.$

- $p = 2$: $\frac{1}{q} + \frac{1}{r} > \frac{1}{2}$

  * $q = 2$, $r \geqslant 2$



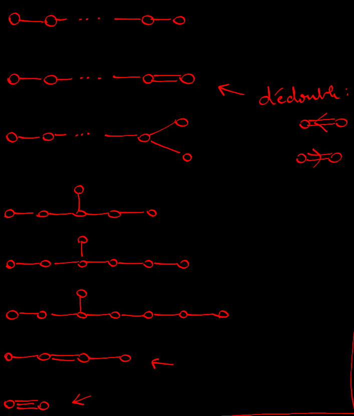
  * $q = 3$, $\frac{1}{r} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ $3 \leqslant r < 6$

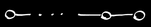
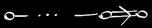
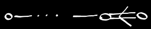
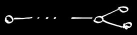
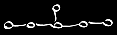
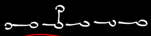
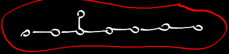
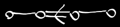
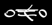


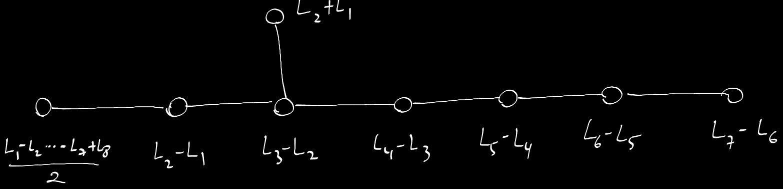
  * $q = 4$: $\frac{1}{r} > \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$ $4 \leqslant r < 2$ ✗

- $p \geqslant 3$ alors $p, q, r \geqslant 3$ $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1$ ✗

Théorème : Les diagrammes admissibles sont exactement :
(de Coxeter)



Nom	Diagramme	Dim	# Racines	$\lvert W \rvert$	Algèbre (réelle comp)
A_n	 $(n \geq 1)$	$n(n+2)$	$n(n+1)$	$(n+1)!$	$su(n+1)$
B_n	 $(n \geq 3)$	$n(2n+1)$	$2n^2$	$n!\,2^n$	$so(2n+1)$
C_n	 $(n \geq 2)$	$n(2n+1)$	$2n^2$	$n!\,2^n$	$sp(n)$
D_n	 $(n \geq 4)$	$n(2n-1)$	$2n(n-1)$	$n!\,2^{n-1}$	$so(2n)$
E_6		78	72	51 840	
E_7		133	126	2 903 040	
E_8		248	240	696 729 600	
F_4		52	48	1152	
G_2		14	12	12	



$\mathbb{R}^8$. $\|L_i\|^2 = \frac{1}{2}$.

Réseau de E_8 : $\left\{ (x_1, \dots, x_8) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^2 \;\middle|\; \sum_{i=1}^{8} x_i \in 2\mathbb{Z} \right\}$.

Les éléments de longueur min (>0) sont :

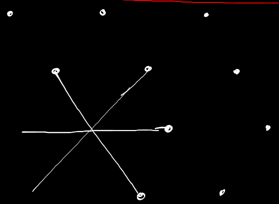
$$\left\{ \begin{array}{lll} (\pm 1, \pm 1, \underbrace{0 \dots 0}_{6}) & \|\cdot\|^2 = 2 & 4 \times \binom{8}{2} = 4\,\frac{8 \cdot 7}{2} = 112 \\ (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) & \|\cdot\|^2 = 8 \times \left(\frac{1}{2}\right)^2 = 2 & \frac{2^8}{2} = 128 \end{array} \right.$$

nombre pair de signes −

Total : $112 + 128 = 240$

- P.S de deux élem $\in \mathbb{Z}$.
- $\|\cdot\|^2 \in 2\mathbb{Z}$
- $\exists$ base de volume 1

Th : Un tel réseau ne peut exister que en dim multiple de 8 !!



Freudenthal

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$Sp(3)$	F_4
$\mathbb{C}$	$SU(3)$	$SU(3)^2$	$SU(6)$	E_6
$\mathbb{H}$	$Sp(3)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Algèbre de Jordan exceptionnelle

$$\begin{pmatrix} \mathbb{R} & \mathbb{O} & \mathbb{O} \\ & \mathbb{R} & \mathbb{O} \\ & & \mathbb{R} \end{pmatrix}$$

$$\mathcal{J} = \{ X \in \mathcal{M}_{3 \times \textcircled{3}}(\mathbb{O}) \;/\; {}^t\bar{X} = X \} \qquad \dim \mathcal{J} = 27$$