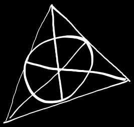
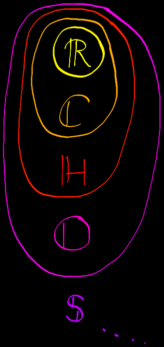
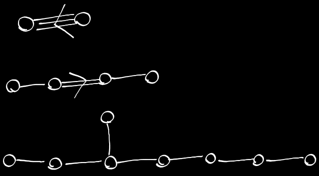
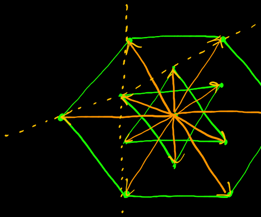


Diagrammes de Dynkin et Objets Exceptionnels







Lie

1880

Killing

1887

Élie Cartan

1894

1886 $\left\{ \begin{array}{l} SO(n) \\ SU(n) \\ Sp(n) \end{array} \right.$

$SO(n)$	Sphères
$SU(n)$	Sphères complexes
$Sp(n)$	" quaternions

Plan :
- Octonions
- Groupe G_2
- "Théorie des graphes" / Géométrie
- Autres groupes exceptionnels.

$$\underbrace{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}} \subset \overset{\text{dim } 1}{\boxed{\mathbb{R}}} \subset \overset{\text{dim } 2}{\boxed{\mathbb{C}}} \subset \overset{\text{dim } 4}{\boxed{\mathbb{H}}} \subset \overset{\text{dim } 8}{\boxed{\mathbb{O}}}$$

a

$i^2 = -1$

$z = a + ib$

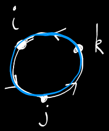
$i^2 = -1$
$j^2 = -1$
$k^2 = -1$
$\underline{ijk = -1}$

$e_1^2 = -1$
$e_2^2 = -1$
$\vdots$
$e_7^2 = -1$

$ij = -ijkk = k$

$ij(kk) = (ijk)k$

associativité



$q = a + ib + jc + kd$

$a, b, c, d \in \mathbb{R}$

	$\mathbb{R}$	$\hookrightarrow \mathbb{C}$	$\hookrightarrow \mathbb{H}$	$\hookrightarrow \mathbb{O}$	$\hookrightarrow \mathbb{S}$
• ordre	✓	✗	✗	✗	✗
• Commutativité $xy = yx$	✓	✓	✗	✗	✗
• associativité $(xy)z = x(yz)$	✓	✓	✓	✗	✗
• alternance $x(xy) = (xx)y$ $x(yy) = (xy)y$	✓	✓	✓	✓	✗

- "Algèbre" réelle de dimension finie : E $\mathbb{R}$-espace vectoriel
  muni d'un produit $E^2 \to E$, $(x,y) \mapsto x \cdot y$ compatible : distributif
  $(\lambda x) \cdot (\mu y) = (\lambda \mu)(x \cdot y)$
  $\lambda, \mu \in \mathbb{R}$, $x, y \in E$

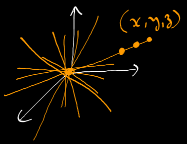
- Algèbre "à division" : Si $a, b \in E$ et $b \neq 0$ alors les équations $a = bx$ et $a = xb$ ont une unique solution.

- Th : (Frobenius) : Associativité $\rightsquigarrow$ $E = \mathbb{R}, \mathbb{C}, \mathbb{H}$.
  (Hurwitz) : Alternance $\rightsquigarrow$ $E = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Plan de Fano :

Plan projectif = ensemble des droites vectorielles dans l'espace de dim 3.

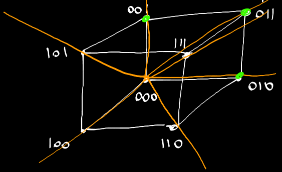
$\mathbb{R}P^2$:



$\mathbb{R}^3$

$$\mathbb{R}P^2 = \{(x,y,z) \in \mathbb{R}^3 - \{(0,0,0)\}\} \Big/ \begin{array}{l}(x,y,z) \\ \equiv (\lambda x, \lambda y, \lambda z) \\ \lambda \in \mathbb{R}^* \end{array}$$

Plan de Fano = plan projectif sur $\left(\mathbb{Z}/2\mathbb{Z} = \{0,1\}\right) = \mathcal{F}$.



$(\mathbb{Z}/2\mathbb{Z})^3$

$|\mathcal{F}| = 7$

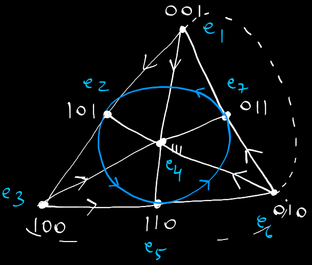
$(e_1 e_2) e_4 = e_3 e_4 = e_7$!!

$e_1 (e_2 e_4) = - e_1 e_6 = - e_7$

Quelles sont les droites dans $\mathcal{F}$?

7

$x = 0$	010	011	001
$y = 0$	100	101	001
$z = 0$	100	110	010
$x + y = 0$	001	110	111
$x + z = 0$	010	101	111
$y + z = 0$	100	011	111
$x + y + z = 0$	110	101	011



$e_1 e_2 = e_3$

$e_1 e_4 = e_5$

$e_1 e_7 = -e_6$

$e_5 e_7 = e_2$

Octonion : $x = \sum_{i=0}^{7} x_i e_i$ $\qquad e_0 = 1$

$y = \sum y_i e_i$

$$xy = (x_0 + x_1 e_1 + \dots + x_7 e_7)(y_0 + y_1 e_1 + \dots)$$
$$= z_0 + z_1 e_1 + \dots + z_7 e_7$$

$$(x, y) = \sum_{i=0}^{7} x_i y_i$$

$$\boxed{G_2 = \{\alpha \in \mathrm{Isom}_{\mathbb{R}}(\mathbb{O}) \mid \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y),\ \forall x, y \in \mathbb{O}\}}$$

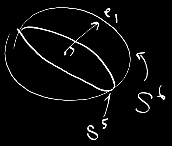
Rq : $\alpha(x\cdot y) = \alpha(x)\cdot\alpha(y) \Rightarrow (\alpha(x), \alpha(y)) = (x, y)$

Donc $\boxed{G_2 \subset SO(7) \subset SO(8)}$

Quelle est la dimension de G_2 ?

$$\mathrm{Dim}(SO(7)) = 6 + 5 + 4 + 3 + 2 + 1 + 0 = 21$$

$$\begin{tikzcd}
e_1 \arrow[dr, rightsquigarrow] & e_2 \arrow[dr, rightsquigarrow] & e_3 \arrow[dr, rightsquigarrow] & e_4 \arrow[dr, rightsquigarrow] & e_5 \arrow[dr, rightsquigarrow] & e_6 \arrow[dr, rightsquigarrow] & e_7 \\
S^6 \arrow[u, Rightarrow] & S^5 \arrow[u, Rightarrow] & S^4 & S^3 & S^2 & S^1 & S^0
\end{tikzcd}$$



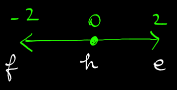
$$\dim(G_2) = \underset{e_1}{6} + 5 + 0 + 3 + 0 + 0 + 0 = 14$$

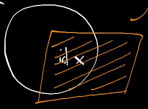
Groupes de Lie, algèbres, racines :

$$\begin{tikzcd}
 & & & \text{Diagramme} \arrow[dll, leftrightarrow] \\
G \arrow[r, rightsquigarrow] & \mathfrak{g} \arrow[r, phantom, "\supset"] & \underset{\text{“Cartan”}}{\mathfrak{h}} \arrow[r, rightsquigarrow] & \text{Système des racines} \arrow[u]
\end{tikzcd}$$

$SO(3)$ $\qquad$ $so(3)$

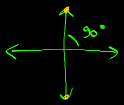
$\mathrm{Vect}\{e, f, h\}$ $\qquad$ $\mathrm{Vect}(h)$

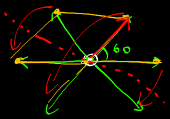




dim = 6

$SO(4)$



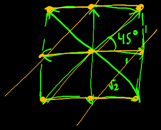


$SU(3)$

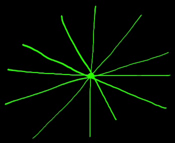
dim = 8

dim = 10

$SO(5)$



G_2



dim = 14 ??

??

θ (degrés)	0°	30°	45°	60°	90°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

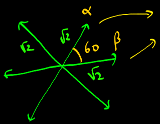
Definition : Un système de racines $R \subset \mathbb{R}^{n}$ est un ensemble *fini* tq :

(1) R génère $\mathbb{R}^n$

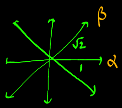
(2) $\forall \alpha \in R$, $-\alpha \in R$ et si $k \in \mathbb{R} - \{1, -1\}$, $k\alpha \notin R$.

(3) R est stable par réflexion $\perp \alpha \in R$.

(4) $\forall \alpha, \beta \in R$, $n_{\beta\alpha} := 2 \dfrac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$



$$2 \frac{\cancel{\|\alpha\|} \cdot \cancel{\|\beta\|} \cdot \cos(\alpha, \beta)}{\cancel{\|\alpha\|} \cdot \cancel{\|\alpha\|}} = 2 \cos 60° = 2 \times \frac{1}{2} = 1$$



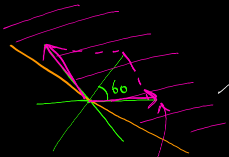
$$n_{\beta\alpha} = 2 \frac{\sqrt{2} \times 1 \times \cos 45}{1 \times 1} = 2\sqrt{2} \, \frac{\sqrt{2}}{2} = 2 \in \mathbb{Z}$$

$$n_{\alpha\beta} = \frac{2 \times 1 \times \sqrt{2} \cos 45}{\sqrt{2} \times \sqrt{2}} = \frac{\cancel{2} \times \cancel{\sqrt{2}}}{\cancel{\sqrt{2}} \times \cancel{\sqrt{2}}} \times \frac{\cancel{\sqrt{2}}}{\cancel{2}} = 1 \in \mathbb{Z}$$

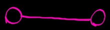
Quels sont les angles possibles dans R ?

$$\mathbb{Z} \ni n_{\alpha\beta} n_{\beta\alpha} = 2^2 \frac{(\alpha, \beta)(\beta, \alpha)}{(\alpha, \alpha)(\beta, \beta)} = 2^2 \frac{\cancel{\|\alpha\|^2} \cancel{\|\beta\|^2}}{\cancel{\|\alpha\|^2} \cancel{\|\beta\|^2}} \cos(\alpha, \beta)^2 = 2 \cos^2(\alpha, \beta) \in \mathbb{Z}.$$

$$\boxed{\cos(\alpha, \beta) = \pm \frac{\sqrt{\text{entier}}}{2}} \leftarrow \text{entre } 0 \text{ et } 4$$

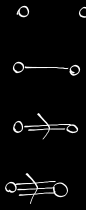


⇝ diagramme de Dynkin.



racines simples

Diagramme de Dynkin : Graphe à n points reliés par :



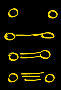
si l'angle est $\frac{\pi}{2}$

" " " $\frac{2\pi}{3}$

" " " $\frac{3\pi}{4}$

$\frac{5\pi}{6}$

Si on retire les flèches, on parle de diagramme de Coxeter.



$\pi/2$

$2\pi/3$

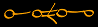
$3\pi/4$

$5\pi/6$

THEOREME (Killing, Cartan, Dynkin, Chevalley, Harish-Chandra, Weyl, Serre)

Il existe une bijection entre :

$\{$algèbres de Lie simples sur $\mathbb{C}\}$ $\longleftrightarrow$ $\{$diagrammes de Coxeter connexes "admissibles"$\}$

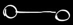


A_{ij}

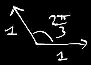
$\{e_i, f_i, h_i\}_{i=1,\dots,4}$

$\mathrm{ad}(e_i)^{1-A_{ij}}(e_j)=0$

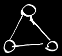
Def : Un diagramme de Coxeter à n nœuds est admissible s'il existe $(e_1,\dots,e_n)\in\mathbb{R}^n$ tq les $\boxed{\|e_i\|=1}$ et les $\boxed{\text{angles}}$ sont donnés par le diagramme



admissible :



✓



non admissible : (exercice)