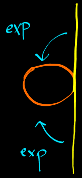


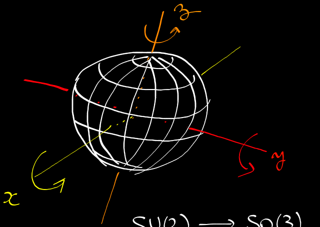
$$t+t^{-1}$$

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

Théorie de Lie

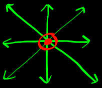
EPISODE II

La symphonie des Sphères



$SU(2) \longrightarrow SO(3)$

$$\log\left(e^X e^Y\right) = X + Y + \tfrac{1}{2}[X,Y] + \dots$$

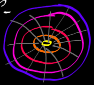


Spin(5)

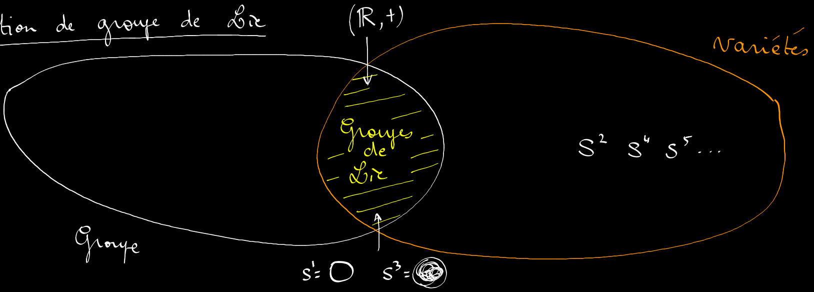
Réels $\mathbb{R}$

Complexes $\mathbb{C}$

Quaternions $\mathbb{H}$



Notion de groupe de Lie



Axiome 1 : les groupes sont intéressants et "utiles"

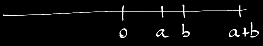
Axiome 2 : les variétés " " " "

Théorème : les groupes de Lie sont très intéressants et très utiles.

Définition : Un groupe de Lie est une variété ($\mathcal{C}^\infty$) (réelle) avec une structure de groupe compatible :

$$\left.\begin{aligned}(a,b) &\longmapsto a\cdot b\\ a &\longmapsto a^{-1}\end{aligned}\right\}\ \text{sont } \mathcal{C}^\infty.$$

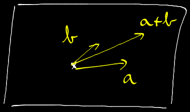
Exemples : • $(\mathbb{R},+)$

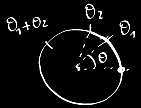


$$(a,b) \mapsto a+b$$

$$a \mapsto -a$$

- $(\mathbb{R}^2,+)$

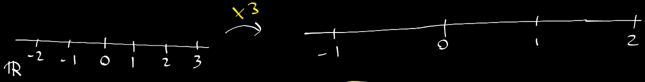


- 

Un groupe $\Longleftrightarrow$ ensemble des symétries d'un objet.

Objet "très" symétrique ⇝ groupe de Lie

Exemple :

- 

Ensemble des automorphismes est $\boxed{(\mathbb{R}^*, \times)} = GL(1,\mathbb{R})$

Groupe de Lie

- L'objet $\mathbb{R}^n$ espace vectoriel. ⇝ $\boxed{\text{ensemble des matrices } n\times n \text{ inversibles} := (GL(n,\mathbb{R}), \times)}$

- L'objet $\mathbb{R}^n$ muni de la distance euclidienne

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\|x\|^2 = x_1^2 + \dots + x_n^2$$

⇝ ensemble des matrices orthogonales de taille $n \times n$

$O(n,\mathbb{R})$

- On peut répéter ceci en remplaçant $\mathbb{R}$ par d'autres ensembles de nombres : $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

réels complexes quaternions.

préserve les

Exploration des sphères

$$S^{n} = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}.$$

dimension de la sphère

$$SO(n) = \{\text{rotations}\}$$
$$= \{ M \in Mat_{n+1,n+1}(\mathbb{R}) \mid \boxed{MM^T = 1\!\!1} \text{ et } \boxed{\det M = 1} \}$$

distances

symétries de S^n

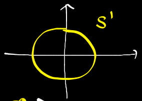
uniquement les rotations.

- **Sphère S^0**. $S^0 = \{ x \in \mathbb{R} \,/\, \|x\| = 1 \} = \{1, \underline{-1}\}$.

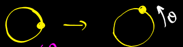
$SO(1) = \{1\}$

L'ensemble des symétries de S^0 est le groupe à deux éléments : $\mathbb{Z}/2\mathbb{Z}$.

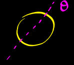
- **Sphère S^1** : $S^1 = \{ x \in \mathbb{R}^2 \,/\, \|x\| = 1 \} =$



Symétries : rotations R_θ



réflexions S_θ



$$O(2) = \{ R_\theta \mid \theta \in [0, 2\pi[\} \cup \underbrace{\cancel{\{ S_\theta \,/\, \theta \in [0, \pi[\}}}$$

$$SO(2) = \{ R_\theta \mid \theta \in [0, 2\pi[\}$$

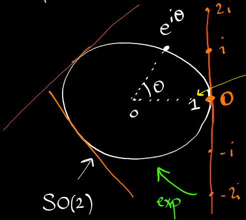
$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \approx \{ z \in \mathbb{C} \mid |z| = 1 \}$$
$$\approx \{ \underline{e^{i\theta}} \,/\, \theta \in \mathbb{R} \}.$$

Groupe abélien (= commutatif).

$$R_\theta R_{\theta'} = R_{\theta'} R_\theta = R_{\theta+\theta'}$$

$$e^{i\theta} \times e^{i\theta'} = e^{i\theta'} \times e^{i\theta} = e^{i(\theta \textcircled{+} \theta')}$$

Exponentielle transforme les sommes en produits (ou presque ...)

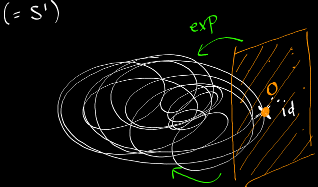


identité du groupe.

$$\exp : (i\mathbb{R}, +) \longrightarrow (SO(2), \times)$$

ADDITION TOUJOURS COMMUTATIF

MULTIPLICATION COMMUTATIF



$$\exp : (\mathfrak{g}, +) \longrightarrow (G, \times)$$

$$X \longmapsto e^X$$

$$\underline{e^X e^Y} \neq e^{X+Y} \overset{\checkmark}{=} e^{Y+X} \neq \underline{e^Y e^X} \quad ???$$

Est-ce que $\log(e^X e^Y) = X + Y$?? $\underline{\text{NON.}}$

Supposons X et Y $\underline{\text{très petits}}$:

$$\begin{cases} e^X = \displaystyle\sum_{n=0}^{\infty} \frac{X^n}{n!} = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \dots \\ \log(1+X) = X - \underline{\tfrac{1}{2}}X^2 + \frac{1}{3}X^3 - \frac{1}{4}X^4 + \dots \end{cases}$$

$$\begin{aligned} \log(e^X e^Y) &= \log\left(\left(1 + X + \frac{X^2}{2} + \dots\right)\left(1 + Y + \frac{Y^2}{2} + \dots\right)\right) \\ &= \log\left(\boxed{1} + \boxed{X + Y + \frac{X^2}{2} + XY + \frac{Y^2}{2}} + \dots\right) \\ &= X + Y + \cancel{\frac{X^2}{2}} + XY + \cancel{\frac{Y^2}{2}} - \frac{1}{2}\left(\cancel{X^2} + \cancel{Y^2} + XY + YX\right) + \dots \\ &= X + Y + \frac{1}{2}XY - \frac{1}{2}YX + \dots \end{aligned}$$

$$\dots = O\begin{pmatrix} X^3 \\ Y^3 \\ XY^2 \\ \vdots \end{pmatrix}$$

$$\boxed{\log(e^X e^Y) = X + Y + \boxed{\frac{1}{2}[X, Y]} + \dots}$$

$$\boxed{e^X e^Y e^{-X} = e^Y + [X, e^Y] + \frac{1}{2}[X, [X, e^Y]] + \frac{1}{6}[X, [X, [X, e^Y]]] + \dots}$$

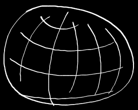
$\updownarrow$

Si on connaît les commutateurs $[\cdot, \cdot]$
alors on connaît le groupe

$$\text{"}\ e^X e^Y = \boxed{e^{[X,\cdot]}}\, e^Y e^X\ \text{"}$$

Théorie des algèbres de Lie

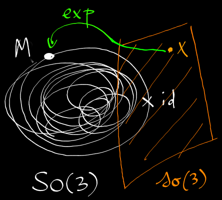
$\boxed{\text{Sphère } S^2}$



$$MBM^T = B$$

$$SO(3) = \left\{ M \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \;\middle|\; MM^T = \mathbb{1} \text{ et } \underline{\det M = 1} \right\}$$

Une rotation $M = e^X$ avec $X \in \underline{so(3)}$



$$\mathbb{1}$$
$$\|$$
$$MM^T = \mathbb{1} = (e^X)(e^X)^T$$
$$= e^X e^{X^T} = \boxed{e^{X+X^T}}$$

$$\boxed{so(3) = \left\{ X \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \;\middle|\; X + X^T = 0 \right\}} \leftarrow \text{espace vectoriel de dimension 3}$$

$$XB + BX^T = 0$$

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

SO(3) est l'ensemble des transfo linéaires qui préservent la forme bilinéaire $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $MM^T = \mathbb{1}$

$$\Longleftrightarrow \underline{MBM^T = B}$$

On change de base et on prend $\underline{B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$.

$$so(3) = \mathrm{Vect}\,(H, X, Y)$$

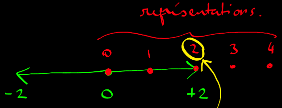
$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H \end{aligned}}$$

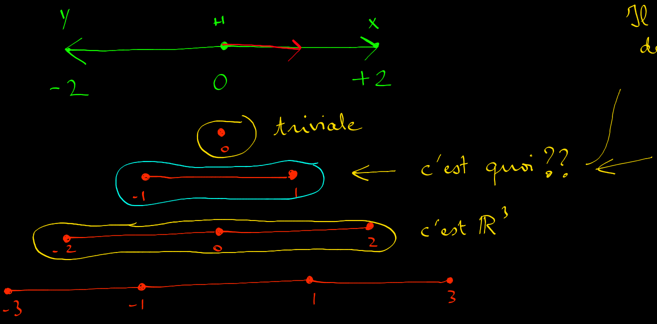
$sl(2, \mathbb{C})$

$\boxed{su(2) \cong so(3)}$

	X	Y
$[H,\cdot]$	+2	−2



représentation de SO(3) de dimension 3 c'est $\underline{\mathbb{R}^3}$!!



Il existerait des objets de dimension 2 sur lesquels SO(3) agit.

$\mathbb{C}^2$ sur lequel agit SU(2).

SPINEUR

$$SO(n) = \{ M \in GL(n,\mathbb{R}) \mid M M^\dagger = \mathbb{1} \quad \text{et} \quad \det M = 1 \}$$

Spécial Orthogonal

$$SU(n) = \{ M \in GL(n,\mathbb{C}) \mid M M^\dagger = \mathbb{1} \quad \text{et} \quad \det M = 1 \}$$

Spécial Unitaire

$$Sp(n) = \{ M \in GL(n,\mathbb{H}) \mid M M^\dagger = \mathbb{1} \}$$

Symplectique

SO(3) : transfo linéaires de $\mathbb{R}^3$ qui préservent $\underline{x^2+y^2+z^2}$

$$x^2+y^2+z^2 = \det \overbrace{\begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix}}^{\xi}$$

$$\boxed{\xi^\dagger = -\xi}$$

Prenons $g \in SU(2)$: $g g^\dagger = \mathbb{1}$

$$\left(\underline{g \xi g^{-1}}\right)^\dagger = (g^\dagger)^{-1} \xi^\dagger g^\dagger = g(-\xi) g^{-1} = -\underline{g \xi g^{-1}}$$

Donc $\varphi_g : \left| \begin{array}{l} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ \xi \longmapsto g \xi g^{-1} \end{array} \right. \in SO(3)$

Donc $\boxed{\begin{array}{l} \varphi : SU(2) \longrightarrow SO(3) \\ \quad g \longmapsto \varphi_g \end{array}}$ est un morphisme de groupes.

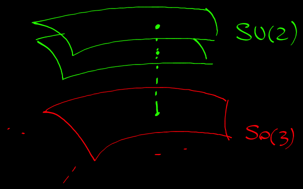
Iso ?

$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SU(2)$ $\qquad \varphi_g = \mathrm{id}_{SO(3)}$

$\mathrm{Ker}(\varphi) = \{\pm \mathrm{id}_{SU(2)}\}$.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1$$

$$SO(3) \cong SU(2)/(\mathbb{Z}/2\mathbb{Z})$$



Double recouvrement

Exemple : $g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in SU(2)$. $\theta : 0 \dashrightarrow 2\pi$

$\varphi_g \in SO(3)$ $\begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix}$ $\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

$$g \xi g^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} ie^{i\theta}z & e^{i\theta}(x+iy) \\ e^{-i\theta}(-x+iy) & -ie^{-i\theta}z \end{pmatrix} \boxed{\begin{pmatrix} iz & e^{2i\theta}(x+iy) \\ e^{-2i\theta}(-x+iy) & -iz \end{pmatrix}}$$

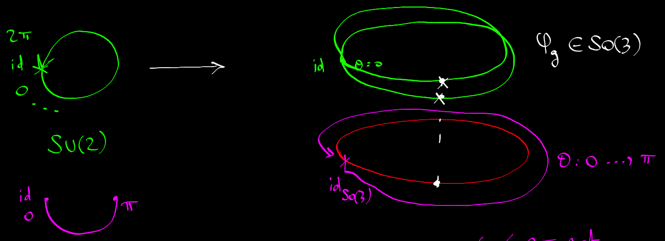
$x \longmapsto \mathrm{Re}\big((\cos 2\theta + i \sin 2\theta)(x+iy)\big) = \cos 2\theta \, x - \sin 2\theta \, y$

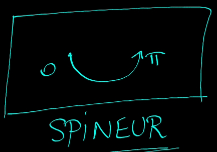
$y \longmapsto \mathrm{Im}\big(\qquad \big) = \sin 2\theta \, x + \cos 2\theta \, y$

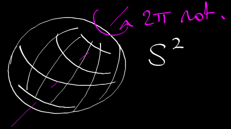
$z \longmapsto \mathrm{Im}(iz) = z$

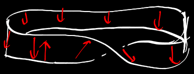
$$\boxed{\varphi_g = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$\theta : 0 \dashrightarrow 2\pi$

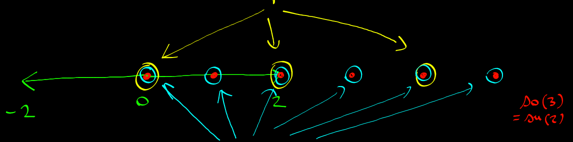








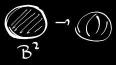
représentations de $SO(3)$: entiers pairs $(j \in \mathbb{N})$

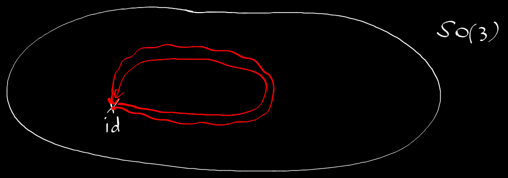


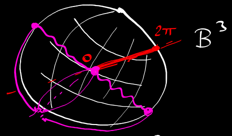
représentations de $SU(2)$.
entiers $\left(j \in \frac{1}{2}\mathbb{N}\right)$

Autre façon de voir : $\boxed{\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}}$.

$$\begin{array}{ccc} & & S^3 \\ & & \| \\ SO(3) & = & SU(2)/(\mathbb{Z}/2\mathbb{Z}) \\ \| & & \\ \mathbb{RP}^3 & & \end{array}$$







$SU(2) = S^3 = B^3/S^2$

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{C}, \; (\;\cdot\;)(\;\cdot\;)^\dagger = \mathbb{1} \text{ et } \det(\;) = 1 \right\}$$

$\approx S^3$. $\qquad \pi_1(SU(2)) = 1$.

$$\boxed{SO(3) = \mathbb{RP}^3 = S^3/(\mathbb{Z}/2\mathbb{Z}) = SU(2)/(\mathbb{Z}/2\mathbb{Z})}$$

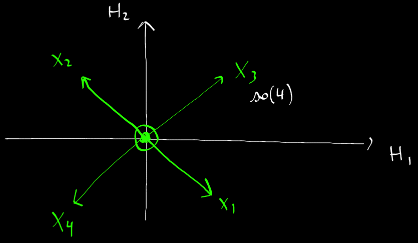
Symétries de S^3 : $SO(4) = \{ M \in GL(4,\mathbb{R}) \mid \det M = 1 \text{ et } M B M^T = B \}$

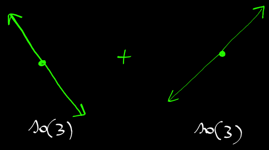
$$B = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

$$so(4) = \{ X \mid XB + BX^T = 0 \}$$

$$= \mathrm{Vect}(H_1, H_2, X_1, X_2, X_3, X_4) \qquad H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

	$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$X_3 = \left(\begin{array}{c\|c} 0 & {}_{-1}{}^{1} \\ \hline 0 & 0 \end{array}\right)$	$X_4 = \left(\begin{array}{c\|c} 0 & 0 \\ \hline {}_{-1}{}^{1} & 0 \end{array}\right)$
$[H_1, \cdot\,]$	1	-1	1	-1
$[H_2, \cdot\,]$	-1	1	1	-1





$$\boxed{so(4) = so(3) \oplus so(3)}$$

$$\underline{w^2 + x^2 + y^2 + z^2} = \det \begin{pmatrix} w + iz & x + iy \\ -x + iy & w - iz \end{pmatrix}$$

$SO(3) = \dfrac{SU(2)}{(\mathbb{Z}/2\mathbb{Z})}$

$SO(4) \simeq \dfrac{SU(2) \times SU(2)}{(\mathbb{Z}/2\mathbb{Z})}$

$g \in \underline{\underline{S}}U(2)$

$g' \in \underline{\underline{S}}U(2)$

$$\underset{\underset{\det 1}{\uparrow}}{g}\, \zeta \, \underset{\underset{\det 1}{\uparrow}}{(g')^{-1}}$$

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow SU(2) \times SU(2) \longrightarrow SO(4) \longrightarrow 1$$

$$q = \begin{pmatrix} w + iz & x + iy \\ -x + iy & w - iz \end{pmatrix} = w + x I + y J + z K$$

$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$$\boxed{\begin{aligned} I^2 &= J^2 = K^2 = -1 \\ IJK &= -1 \end{aligned}}$$

Quaternions.

$$\boxed{Sp(1) = SU(2) = Spin(3)}$$

$$\boxed{Sp(1) \times Sp(1) = Spin(4).}$$

$$IJ = -IJKK = K$$

$$q = w + xI + yJ + zK$$

$$\bar q = w - xI - yJ - zK$$

$$|q|^2 = q\bar q = w^2 + x^2 + y^2 + z^2$$

$SO(n)$ $\quad Spin(n)$ $\quad (n \geq 3)$ $\qquad n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$

$SU(n)$ $\qquad 2n^2 - n^2 - 1 = n^2 - 1$

$Sp(n)$ $\qquad 4n^2 - n - 4\,\frac{n(n-1)}{2} = 2n^2 + n.$

$$\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} & \cdot & \end{pmatrix}^{\dagger} = \begin{pmatrix} \boxed{q_1\bar q_1 + q_2\bar q_2} & \boxed{q_1\bar q_3 + q_2\bar q_4} \\ q_3\bar q_1 + q_4\bar q_2 & q_3\bar q_3 + q_4\bar q_4 \end{pmatrix}$$

n	1	2	3	4	5	6	7
$Spin(n)$ $SO(n)$	0	1 Ab	3 $\xrightarrow{\times 2}$	6	10	15	21
$SU(n)$	0	3	8	15	24	35	
$Sp(n)$	3	10	21	36	55	78	

G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

Sphère S^4 : $\quad SO(5) = \{\ldots\}$

$$so(5) = \{\ldots\} \qquad B = \left(\begin{array}{c|c|c} 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ \hline \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

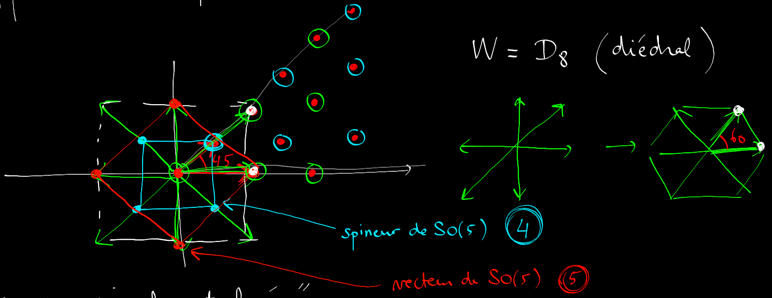
$$so(5) = \mathrm{Vect}\,(H_1, H_2, X_1, X_2, X_3 \ldots X_8)$$

$$= \left\{ \left(\begin{array}{cccc|c} h_1 & x_1 & 0 & x_3 & x_5 \\ x_2 & h_2 & -x_3 & 0 & x_6 \\ 0 & x_4 & -h_1 & -x_2 & x_7 \\ -x_4 & 0 & -x_1 & -h_2 & x_8 \\ \hline -x_7 & -x_8 & -x_5 & -x_6 & 0 \end{array}\right) \right\}$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

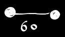
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
$[H_1, .]$	+1	-1	1	-1	1	0	-1	0
$[H_2, .]$	-1	+1	1	-1	0	1	0	-1

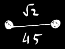
$W = D_8$ (diédral)



"algèbre non simplement lacée"

S^5 : $SO(6) = SU(4) / (\mathbb{Z}/2\mathbb{Z})$ $\underline{\mathrm{Spin}(6) = SU(4)}$





Conclusion :

- Ça devient compliqué … il faut plus de "théorie" pour comprendre (Episode III).
- Isomorphismes exceptionnels :

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathrm{Spin}(3) =$	$SU(2) =$	$Sp(1)$
$\mathrm{Spin}(4) =$	$SU(2)^2 =$	$Sp(1)^2$
$\mathrm{Spin}(5) =$		$Sp(2)$
$\mathrm{Spin}(6) =$	$SU(4)$	
⋮		

Quelle est la logique ?

- Qu'en est-il des octonions ? (Groupes exceptionnels : 5) }
- Est-ce qu'on peut classifier ? (Vue globale du musée) (~ oui) (Simples)

Knapp : Lie Groups beyond an introduction ←

Fulton - Harris : Representation theory. ←

|| Zuber : notes sur algèbres de Lie .. ←

Humphreys