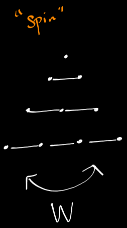
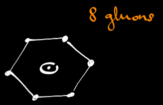


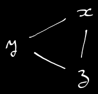
Théorie de Lie

EPISODE I

Des racines et des poids



$$\mathrm{ad}(X)(Y) = [X, Y]$$



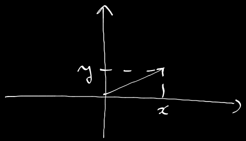
$$\frac{\sin(nx)\sin(mx)}{\sin(x)} = \sin((n+m)x) \overset{-2}{\curvearrowright} + \cdots \overset{-2}{\curvearrowright} + \sin((n-m)x)$$

$$2 \otimes 2 = 3 \overset{-2}{\curvearrowright} \oplus 1 \qquad 3 \otimes \overline{3} = 8 \oplus 1$$

Exemple : $sl(2, \mathbb{C})$

Matrices de taille 2x2 à coefficients dans $\mathbb{C}$.

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle/ a, b, c, d \in \mathbb{C} \right\}$$



$$\begin{array}{cc} & \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \end{array}$$

$$a+b=b+a$$

Structures algébriques :

$(E, +)$ groupe abélien (comm.)

$(E, +, \times)$ anneau (en général non commut.)

$(E, +, \cdot)$ espace vectoriel

$(E, +, \times, \cdot)$ algèbre.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$$

$$k \in \mathbb{C} \qquad k \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

X et Y matrices. $XY \neq YX$

Algèbre de Lie : $(g, +, [\,,\,], \cdot)$ est une algèbre de Lie

crochet de Lie

si c'est une algèbre telle que :

$$\begin{cases} [X,Y] = -[Y,X] \\ [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \end{cases} \qquad \forall X,Y,Z \in g$$

Exemple : l'ensemble des matrices 2x2 muni du commutateur

$$[X,Y] = XY - YX.$$

Sous-ensemble : $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \ a \in \mathbb{C} \right\}$. $\quad tr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d.$

$$\left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & a' \end{pmatrix} \right] = 0$$

$SL(2, \mathbb{C})$

det = 1

$$\boxed{sl(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \ a, b, c \in \mathbb{C} \right\}}$$

muni du crochet de Lie commutateur.

Trouvons une base de $sl(2, \mathbb{C})$: $\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{H}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{X}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{Y} \right\}$

$$\boxed{\begin{aligned} [H,X] &= 2X \\ [H,Y] &= -2Y \\ [X,Y] &= H \end{aligned}}$$

← idée abstraite de l'algèbre de Lie.

Trouver 3 objets $\{H, X, Y\}$ qui vérifient ces relations, c'est trouver une REPRESENTATION de cette algèbre

Exemples de représentations :

	Y	H	X
Fondamentale	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Triviale	0	0	0
Polynômes homo de degré 2	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
Adjointe	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Poly hom deg n	$-x\frac{\partial}{\partial y}$	$-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$-y\frac{\partial}{\partial x}$

Espace des polynômes homogènes de degré 1 en x et y.

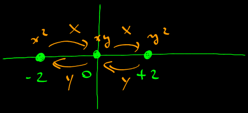
$$\{ a\underline{\underline{x}} + b\underline{\underline{y}} \mid a, b \in \mathbb{C} \}$$

$$X: \begin{array}{l} x \to 0 \\ y \to x \end{array} \qquad Y: \begin{array}{l} x \to y \\ y \to 0 \end{array} \qquad H: \begin{array}{l} x \to x \\ y \to -y \end{array}$$

Espace des polynômes homogènes de degré n en x et y.

$$X = -y\frac{\partial}{\partial x} \qquad Y = -x\frac{\partial}{\partial y} \qquad H = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$\begin{aligned} HP &= [X,Y]P \\ &= -y\frac{\partial}{\partial x}\left(-x\frac{\partial}{\partial y}P\right) - \left(-x\frac{\partial}{\partial y}\left(-y\frac{\partial}{\partial x}P\right)\right) \\ &= +y\frac{\partial}{\partial y}P + \cancel{yx\frac{\partial^2}{\partial x\partial y}P} - x\frac{\partial}{\partial x}P - \cancel{xy\frac{\partial^2}{\partial x\partial y}P} \\ &= \left(y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right)P \end{aligned}$$



$\underline{\underline{n=2}}$

$$X: \begin{cases} \boxed{x^2} \longrightarrow \boxed{-2x\underline{y}} \\ \boxed{xy} \longrightarrow \boxed{-y^2} \\ y^2 \longrightarrow 0 \end{cases} \qquad Y: \begin{cases} x^2 \longrightarrow 0 \\ xy \longrightarrow -x^2 \\ y^2 \longrightarrow -2xy \end{cases}$$

$$\begin{array}{l} \alpha = -2 \\ Y \circlearrowleft \quad H: \begin{cases} x^2 \to \underline{-2}\,x^2 \; X \\ xy \to \underline{0} \quad \alpha = 0 \\ y^2 \to \underline{+2}\,y^2 \; X \end{cases} \\ Y \circlearrowleft \\ \alpha = +2 \end{array}$$

$$\begin{array}{ccc} x^2 & xy & y^2 \\ \hline \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Représentation "adjointe".

On fait agir la rep fondamentale par le crochet de Lie :

$$\operatorname{ad}(X)(M) = [X, M]$$

$$\operatorname{ad}X: \begin{cases} X \to 0 \\ Y \to H \\ H \to -2X \end{cases} \qquad \operatorname{ad}Y: \qquad \operatorname{ad}H:$$

Classification : Une représentation de dimension finie de $\mathfrak{sl}(2,\mathbb{C})$ est un $\rho : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(V)$ avec V espace vectoriel de dimension finie tel que $\rho([X,Y]) = [\rho(X),\rho(Y)]$ $\forall X,Y \in \mathfrak{sl}(2,\mathbb{C})$.

On veut classifier les rep. irréductibles.

Not : si $X \in \mathfrak{g}$, $v \in V$,
$X \cdot v = \rho(X)(v)$

Soit V une irrep. L'action de H est diagonalisable.

$$V = \bigoplus \text{Espaces propres de } \rho(H)$$

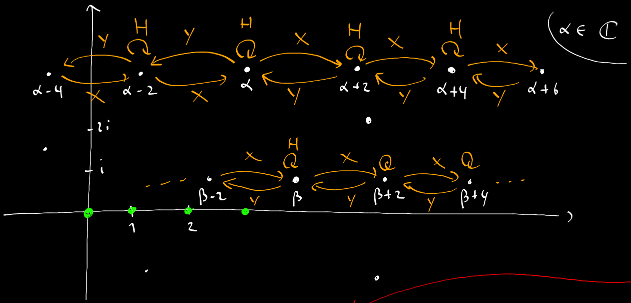
$$V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha \qquad V_\alpha = \{ v \in V \mid H \cdot v = \alpha v \}.$$

Calcul fondamental : soit $v \in V_\alpha$. Alors : $H \cdot v = \alpha v$

$$\begin{aligned} H \cdot \underbrace{X \cdot v} &= [H,X]\cdot v + X \cdot H \cdot v \\ &= 2\,X\cdot v + \alpha\, X \cdot v \\ &= (\alpha + 2)\, \underbrace{X \cdot v} \end{aligned}$$

$$\boxed{Y \cdot v \in V_{\alpha-2}}$$

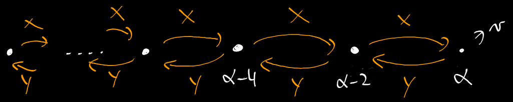
$$\boxed{X \cdot v \in V_{\alpha+2}}$$

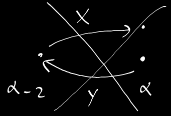


fini !!

Graphe Connexe !!

Il faut 1 seule chaîne, de longueur finie !





Lemme. Soit $v \in V_\alpha$ (α = "plus grand")

$\{v, Y\cdot v, Y\cdot Y\cdot v, \dots\}$ engendre V.

Preuve. Soit $W = \langle v, Y\cdot v, Y\cdot Y\cdot v, \dots \rangle$

On montre que W est stable par $\textcircled{X}$, Y, H. trivial

- $X\cdot v \in V_{\alpha+2} = \{0\}$ $\qquad X\cdot v = 0 \in W$
- $\underline{X\cdot Y\cdot v} = [X,Y]\cdot v + Y\cdot \cancel{X\cdot v}^{\,0}$
  $= H\cdot v = \alpha v \in W$
- $X\cdot Y\cdot Y\cdot v = [X,Y]\cdot Y\cdot v + Y\,\underbrace{(X\cdot Y\cdot v)}_{}$ $\in W$
  $= H\cdot Y\cdot v + Y\alpha v$
  $= ((\alpha-2)+\alpha)\,Y\cdot v$

$\vdots$

- $X\cdot Y^m\cdot v = (\alpha + (\alpha-2) + (\alpha-4) + \dots)\, Y^{m-1}\cdot v$
  $= m(\alpha - m + 1)\, Y^{m-1} v.$

m minimal tel que $Y^m\cdot v = 0$

$$\boxed{\begin{array}{l} m \in \mathbb{N} \\ m \geq 1 \end{array}}$$

$$X\cdot Y^m\cdot v = 0 = \underset{\neq 0}{m}\,\underbrace{(\alpha - m + 1)}_{= 0}\,\underbrace{Y^{m-1} v}_{\neq 0}$$

$$\boxed{\alpha = m - 1 \in \mathbb{N}}$$

Spin $\in \frac{1}{2}\mathbb{N}$

Conclusion :

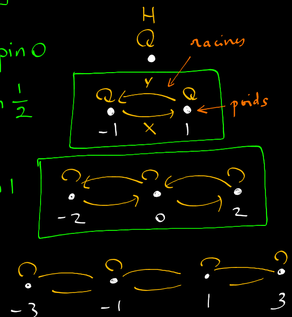
a^+

a

J_x

J_y

J_z



$\alpha = 0$ Spin 0 $[0]$ dim = 1

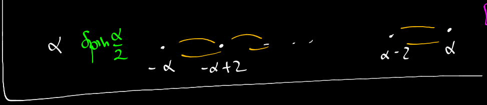
$\alpha = 1$ Spin $\frac{1}{2}$ $[1]$ dim = 2

$\alpha = 2$ Spin 1 $[2]$ dim = 3

$\alpha = 3$ Spin $\frac{3}{2}$ $[3]$ dim = 4

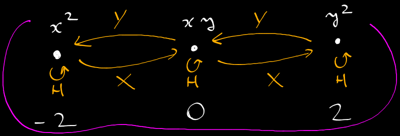
$\vdots$

J_+ J_-



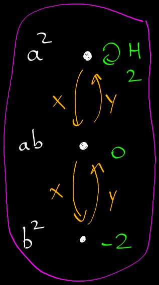
$[\alpha]$ $\underline{\dim = \alpha+1}$

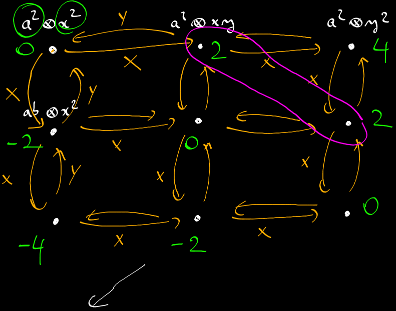
Poly degré 2 homo :

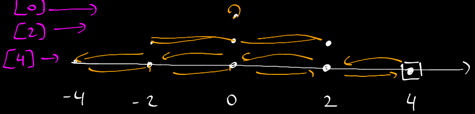


= [2]

[2] =







$$\left(\frac{\sin 3x}{\sin x}\right)^2 = \frac{\sin 5x}{\sin x} + \frac{\sin 3x}{\sin x} + \frac{\sin 1x}{\sin x}$$

$[2] \otimes [2] = [4] \oplus [2] \oplus [0] \longrightarrow$

$$\boxed{\frac{(\sin 3x)^2}{\sin x} = \sin 5x + \sin 3x + \sin x}$$

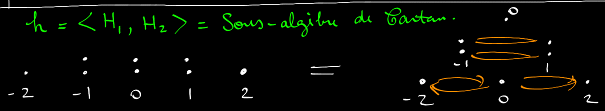
$3 \otimes 3 = 5 \oplus 3 \oplus 1 \Longleftarrow$

Exemple 2 : $\mathfrak{sl}(3,\mathbb{C})$

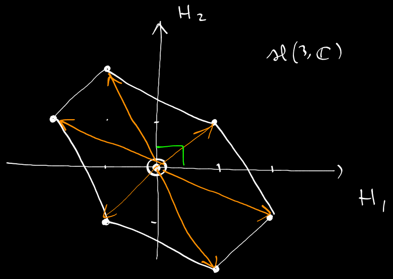
$\left(X = (\ 0\ 0) \quad Y = (\ 1\ 0) \quad H = (\ 0\ -1) \right)$

	H_1	H_2	X			Y		
	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$
$\mathrm{ad}\,H_1$	0	0	+2	+1	−1	−2	−1	+1
$\mathrm{ad}\,H_2$	0	0	−1	+1	+2	+1	−1	−2

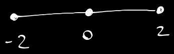
$h = \langle H_1, H_2 \rangle$ = Sous-algèbre de Cartan.



$\mathfrak{sl}(3,\mathbb{C})$



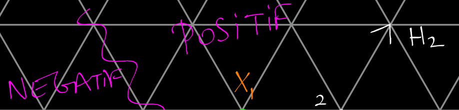
$\mathfrak{sl}(2,\mathbb{C})$

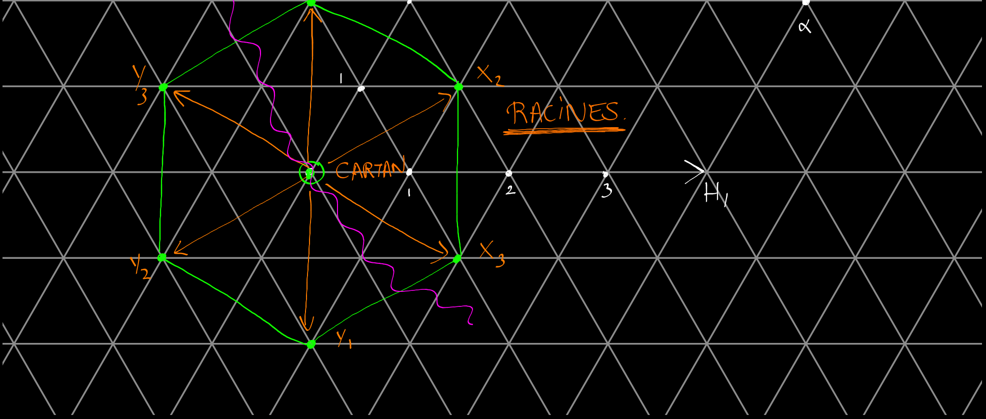


$\alpha \in h^*$

$\alpha : h \to \mathbb{C}$

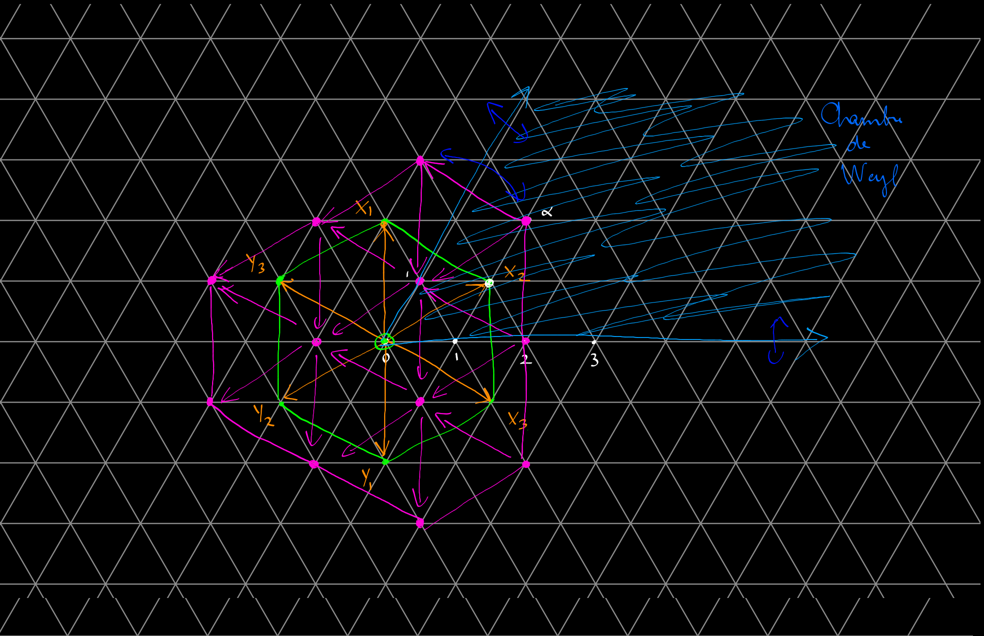
$$V = \bigoplus_{\alpha \in h^*} V_\alpha \qquad V_\alpha = \{ v \in V \mid \forall H \in h,\ H \cdot v = \alpha(H)\, v \}$$

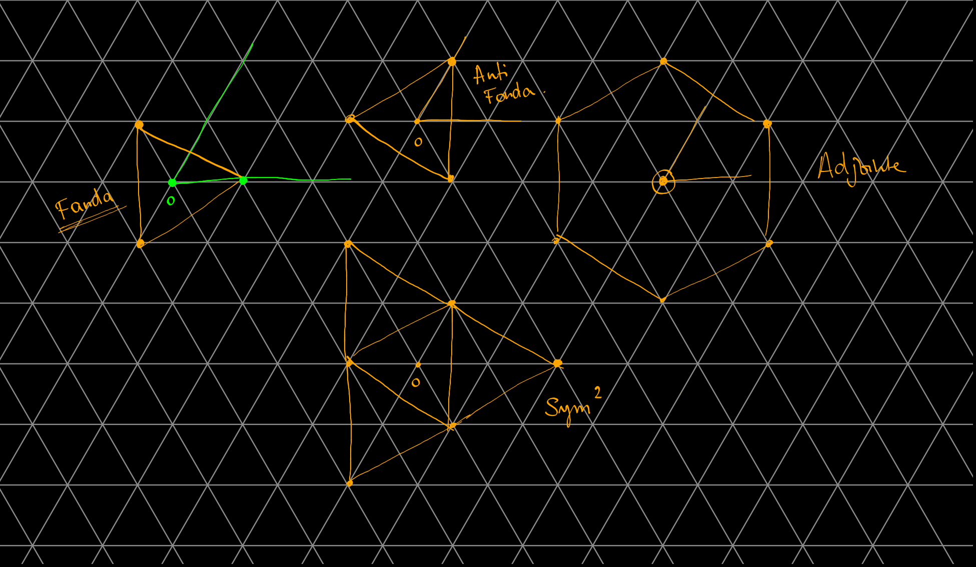




L'algèbre de Lie $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ se décompose en

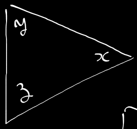
$$\mathfrak{g} = \underset{\substack{\uparrow \\ H_1,\ H_2}}{\mathfrak{h}} \oplus \underset{\substack{\alpha\in\Delta \\ \uparrow \\ \text{racines}}}{\bigoplus} \mathfrak{g}_\alpha \qquad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h},\ [H,X] = \overbrace{\alpha(H)}^{\in\mathbb{C}} X\}$$

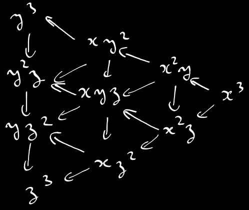




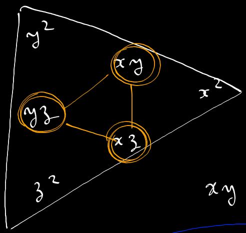
Polynômes homogènes de degré ① en 3 variables : x y z

(n)



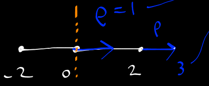


Degré 2 :



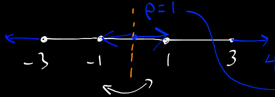
$xy = yx$!

Caractère .



$$\boxed{t^{-2}+1+t^{2}} = \frac{t^{③} - t^{\text{(-3)}}}{t^{①} - t^{\text{(-1)}}}$$

Niveau (3ᵉ)



$$\ldots = \frac{t^{④} - t^{\text{(-4)}}}{t^{①} - t^{\text{(-1)}}}$$

S_2

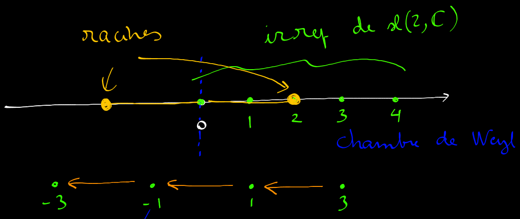
$$\boxed{\frac{\displaystyle\sum_{w\in S_2} t^{w(\alpha+\rho)}\varepsilon(w)}{\displaystyle\sum_{w\in S_2} t^{w(\rho)}\varepsilon(w)} = \underline{\text{caractère}}.}$$

$\mathfrak{sl}(2)$

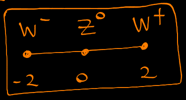
$$\boxed{\frac{\displaystyle\sum_{w\in S_3} e^{w(\alpha+\rho)(z)}\varepsilon(w)}{\displaystyle\sum_{w\in S_3} e^{w(\rho)(z)}\varepsilon(w)} = \text{Caractère} = \sum_{\underline{\text{poids}}} e^{z}.}$$

Conclusion.

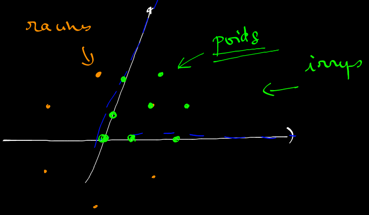
$\mathfrak{sl}(2,\mathbb{C})$



$2 \otimes 2 = 3 \oplus 1$



$\mathfrak{sl}(3,\mathbb{C})$

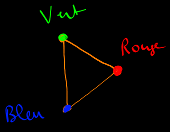


Algèbre de Lie du groupe $SL(3,\mathbb{C})$

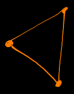
complexification du $SU(3)$

Groupe de jauge de l'interaction forte.

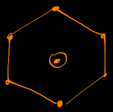
$3 \otimes \bar{3} = \underline{8 \oplus 1}$ $\neq$ Voie octuple



Quarks



Anti Quarks



Gluons.

Chromodynamique.

$3 \otimes \bar{3} \otimes 8 = \ldots\ldots \quad \oplus \textcircled{1}$