## Problem 1 Vector Operators

1. Consider a classical vectorial quantity $\mathbf{V} = (V_x, V_y, V_z)$. Show that under a rotation by a small angle $\psi$ around a unit vector $\mathbf{u}$, the components of $\mathbf{V}$ transform as $V_\alpha \to V_\alpha + \psi \epsilon_{\alpha\beta\gamma} u_\beta V_\gamma$.
2. We want to find the quantum mechanical analogue of this transformation. Consider a state $|\varphi\rangle$ in a Hilbert space, and a vectorial operator $\hat{\mathbf{V}} = (\hat{V}_x, \hat{V}_y, \hat{V}_z)$. Show that the expectation value $\langle \hat{V}_\alpha \rangle$ transforms as
$$\langle \hat{V}_\alpha \rangle \to \langle \hat{V}_\alpha \rangle + i\psi u_\beta \langle [\hat{L}_\beta, \hat{V}_\alpha] \rangle \tag{1}$$
under such a rotation, where $\hat{\mathbf{L}}$ is the angular momentum operator.
3. Deduce the commutator $[\hat{L}_\alpha, \hat{V}_\beta]$.

*Solution to Problem 1*:

1. One can derive this formula from a variety of points of view. An explicit expression for the rotation of a three-dimensional vector is given by Rodrigues' rotation formula, which states that $\mathbf{V}$ is transformed to
$$\mathbf{V} \to (\cos\psi)\mathbf{V} + (\sin\psi)(\mathbf{u} \times \mathbf{V}) + (1 - \cos\psi)\mathbf{u}(\mathbf{u} \cdot \mathbf{V})\,. \tag{2}$$
Expanding at first order in $\psi$ one gets
$$\mathbf{V} \to \mathbf{V} + \psi(\mathbf{u} \times \mathbf{V})\,, \tag{3}$$
which gives the result.
2. Rotations in the Hilbert space are realized by the unitary operator
$$\hat{U} = \exp(-i\psi \mathbf{u} \cdot \hat{\mathbf{L}}) = 1 - i\psi \mathbf{u} \cdot \hat{\mathbf{L}} + o(\psi)\,, \tag{4}$$
which means that the state $|\varphi\rangle$ is mapped to $\hat{U}|\varphi\rangle$. The expectation value of $\mathbf{V}$ in this state then transforms as
$$\langle \hat{\mathbf{V}} \rangle = \langle \varphi | \hat{\mathbf{V}} | \varphi \rangle \to \langle \varphi | \hat{U}^\dagger \hat{\mathbf{V}} \hat{U} | \varphi \rangle\,. \tag{5}$$
At first order in $\psi$ the right hand side is
$$\langle \varphi | \hat{U}^\dagger \hat{\mathbf{V}} \hat{U} | \varphi \rangle = \langle \varphi | (1 + i\psi \mathbf{u} \cdot \hat{\mathbf{L}}) \hat{\mathbf{V}} (1 - i\psi \mathbf{u} \cdot \hat{\mathbf{L}}) | \varphi \rangle = \langle \varphi | \hat{\mathbf{V}} | \varphi \rangle + \langle \varphi | i\psi [\mathbf{u} \cdot \hat{\mathbf{L}}, \hat{\mathbf{V}}] | \varphi \rangle\,. \tag{6}$$
3. Identifying the two results, one find
$$[\hat{L}_\alpha, \hat{V}_\beta] = i\epsilon_{\alpha\beta\gamma} \hat{V}_\gamma\,. \tag{7}$$

## Problem 2 Tensor product decomposition

You saw that for each $l$ there exist an (a priori complex) irreducible representation $\mathcal{E}_l$ of $SU(2)$ of dimension $2l + 1$, spanned by vectors $|l, m\rangle$, $m \in \{-l, -l + 1, ..., l - 1, l\}$. It is usually called spin-$l$-representation. We can identify $\mathcal{E}_l \cong \mathbb{C}^{2l+1}$.

1. Using Clebsch-Gordan theory, decompose $\mathcal{E}_1 \otimes \mathcal{E}_1$ and $\mathcal{E}_{1/2} \otimes \mathcal{E}_{1/2}$ into irreducible representations of $SU(2)$.
2. We would like to understand how this decomposition exactly works. Consider the defining (3-dim.) representation of $SO(3)$ on $\mathcal{E}_1$, $\rho(R)v = Rv$ where $R \in SO(3)$ and $v \in \mathbb{R}^3$ (in odd dimensions, the (a priori complex) spin-$k$ representations can be made real). Think of the space $\mathcal{E}_1 \otimes \mathcal{E}_1$ as $3 \times 3$ matrices that can be decomposed into symmetric and anti-symmetric subspaces. Show that the tensor product representation of the defining representation of $SO(3)$, $\rho \otimes \rho$, leaves them invariant and identify the remaining 1-dim. subspace that can be extracted from the symmetric subspace, such that all subspaces are irreducible under $SO_3$.
3. Consider the defining (2-dim.) representation of $SU(2)$ on $\mathcal{E}_{1/2}$ and decompose $\mathcal{E}_{1/2} \otimes \mathcal{E}_{1/2}$ into irreducible subspaces as above.

*Solution to Problem 2*:

1. Using the general formula for the decomposition of tensor products of spin-$l$ representations of $SU(2)$, $\mathcal{E}_{l_1} \otimes \mathcal{E}_{l_2} = \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} \mathcal{E}_l$, we find, $\mathcal{E}_1 \otimes \mathcal{E}_1 = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{E}_{1/2} \otimes \mathcal{E}_{1/2} = \mathcal{E}_0 \oplus \mathcal{E}_1$.
2. The space $\mathcal{E}_1 \otimes \mathcal{E}_1 = \mathbb{R}^3 \otimes \mathbb{R}^3$ is isomorphic to $\mathrm{Mat}(3, \mathbb{R})$ by the identification $v \otimes w \to vw^T$ where $v, w \in \mathbb{R}^3$. On this space, the tensor product representation $\rho^2 := \rho \otimes \rho$ of $SO(3)$ acts as $\rho^2(R)A = RAR^T$, $A \in \mathrm{Mat}(3, \mathbb{R})$, $R \in SO(3)$.

   Let $A = A^T$ be an (anti-)symmetric matrix in $\mathrm{Mat}(3, \mathbb{R})$, then it is easy to see that $\rho^2(R)A$ is also (anti-)symmetric. The symmetric subspace is 6-dimensional and the antisymmetric subspace is 3-dimensional. Comparing with the former question, we therefore know that there must be a further invariant subspace of dimension 1, hidden in the space of symmetric matrices. This is the trace of $A$. Let $\mathrm{tr}(A) = 0$ and $A$ symmetric. Then also $\mathrm{tr}(\rho^2(R)A) = \mathrm{tr}(RAR^T) = \mathrm{tr}(A) = 0$ since $R^T R = \mathbb{I}_3$. To sum up, any real $3x3$ matrix $A$ decomposes as

$$A = \frac{1}{2}(A + A^T - \frac{1}{3}\mathrm{tr}(A)) + \frac{1}{2}(A - A^T) + \frac{1}{6}\mathrm{tr}(A)\mathbb{I}_3 \tag{8}$$

3. We do an analogue identification of $\mathcal{E}_{1/2} \otimes \mathcal{E}_{1/2} = \mathbb{C}^2 \otimes \mathbb{C}^2$ with $\mathrm{Mat}(2, \mathbb{C})$ by $v \otimes w \to vw^T$ where $v, w \in \mathbb{C}^2$. Note that $v \otimes w \to vw^\dagger$ does not conserve linearity of the tensor product and therefore does not provide an isomorphism to complex $2x2$ matrices. As above, the tensor product representation acts as $\rho^2(U)A = UAU^T$, $U \in SU(2)$, $A \in \mathrm{Mat}(2, \mathbb{C})$ and the decomposition into symmetric and antisymmetric subspaces works analogously. But notice that we can no longer subtract the trace, since $\mathrm{tr}(\rho^2(U)A) = \mathrm{tr}(UAU^T) \neq \mathrm{tr}(A)$.

## Problem 3 Magnetic moment and gyromagnetic factor

Consider a quantum system $\mathcal{S}$ described by a Hilbert space $\mathcal{H}$. We call $\hat{\boldsymbol{L}}$ its total angular momentum (including spin). In the presence of a magnetic field $\boldsymbol{B}$, we write its Hamiltonian as $\hat{H}(\boldsymbol{B})$. For small fields, we expand $\hat{H}$ as

$$\hat{H}(\boldsymbol{B}) = \hat{H}_0 - \sum_\alpha \hat{M}_\alpha B_\alpha + \dots \tag{9}$$

where, by definition, $\hat{H}_0 = \hat{H}(0)$ and $\hat{M}_\alpha = -\partial_{B_\alpha} H(\boldsymbol{B} = 0)$.

1. Assuming that the total system (system and field) is rotationally symmetric, show that $\hat{H}_0$ and $\hat{\boldsymbol{M}}$ are respectively scalar (rank 0) and vector (rank 1) tensor. A rank $k$ tensor is an is an element of the vector space of operators on Hilbert space that transforms in a spin-$k$ representation of $\mathfrak{su}(2)$ or $SU(2)$. *Hint:* Consider the effect of infinitesimal rotations on $\boldsymbol{B}$ and $\hat{H}$.

2. We consider first the case of a zero external magnetic field. Let us decompose the system Hilbert space into eigenstates of the angular momentum operator $\hat{L}_z$ as $|n, \ell, m_\ell\rangle = |n\rangle \otimes |\ell, m_\ell\rangle$, where $|l, m_l\rangle$ forms for each $l$ an irreducible $\mathfrak{su}(2)$ representation. Here $n$ labels additional degrees of freedom, that are not touched by rotations, and it is chosen such that $\hat{H}_0$ is diagonal when acting of $|n\rangle$. Show that the eigenstates of $\hat{H}_0$ are also of the form $|n, \ell, m_\ell\rangle$. Using Winger Eckart's theorem, show that the corresponding energies are of the form $E_{n,\ell}$, i.e. independent of $m_l$.

3. We now consider the case of a weak (albeit finite) magnetic field. Using the Wigner Eckart theorem, show that, at first order in perturbation, the energy spectrum can be interpreted by endowing the system with a magnetic moment $\hat{\boldsymbol{M}}_{n\ell} = \gamma_{n\ell}\hat{\boldsymbol{L}}$.

*Solution to Problem 3*:

Rotational symmetry of the full system tells us that a rotation of the full Hamiltonian should have no impact, if we also rotate the magnetic field accordingly.

We describe rotations of the Hamiltonian by $\hat{H}_\theta = \hat{U}^\dagger_{\boldsymbol{n}}(\theta)\hat{H}\hat{U}_{\boldsymbol{n}}(\theta)$ (this can be interpreted as a Heisenberg picture description of the change of our coordinate system or an inverse rotation of $\hat{H}$). The unitary operator $\hat{U}_{\boldsymbol{n}}(\theta)$ is generated by the angular momentum operator $\hat{\mathbf{L}}$ and for small $\theta$ we expand as

$$\hat{U}_{\boldsymbol{n}}(\theta) = \hat{\mathbb{I}} - i\theta \sum_{j=1}^{3} n_j \hat{L}_j + \mathcal{O}(\theta^2), \tag{10}$$

where $\mathbf{n} = (n_1, n_2, n_3)^T$ is the rotation axis and $\theta$ the angle. On the other hand, rotations of the magnetic field vector $\boldsymbol{B} \in \mathbb{R}^3$ are described by

$$R_{\boldsymbol{n}}(\theta)\boldsymbol{B} = \boldsymbol{B} + \theta\boldsymbol{n} \times \boldsymbol{B} + \mathcal{O}(\theta^2). \tag{11}$$

Symmetry now requires that

$$\hat{U}^\dagger_{\boldsymbol{n}}(\theta)\hat{H}(R_{\boldsymbol{n}}(\theta)\boldsymbol{B})\hat{U}_{\boldsymbol{n}}(\theta) = \hat{H}(\boldsymbol{B}). \tag{12}$$

1. Let us first consider the case $\boldsymbol{B} = \mathbf{0}$ with $\hat{H}(\mathbf{0}) = \hat{H}_0$. We obtain

$$\hat{U}^\dagger_{\boldsymbol{n}}(\theta)\hat{H}_0\hat{U}_{\boldsymbol{n}}(\theta) = \hat{H}_0 + i\theta \sum_{j=1}^{3} n_j[\hat{L}_j, \hat{H}_0] + \mathcal{O}(\theta^2). \tag{13}$$

Symmetry requires the right-hand side to be equal to $\hat{H}_0$ for arbitrary $\boldsymbol{n}$. So all nonzero orders in $\theta$ must vanish, leading to the condition

$$[\hat{L}_j, \hat{H}_0] = 0, \qquad j = 1, 2, 3. \tag{14}$$

Hence, $\hat{H}_0$ is a scalar operator.

For the case $\boldsymbol{B} \neq \boldsymbol{0}$, we only have to consider the part $\hat{\boldsymbol{M}} \cdot \boldsymbol{B}$ since we already know that $\hat{H}_0$ is a scalar operator. We obtain after the rotation

$$\begin{aligned}\hat{U}^\dagger_{\boldsymbol{n}}(\theta) \left[\hat{\boldsymbol{M}} \cdot (R_{\boldsymbol{n}}(\theta)\boldsymbol{B})\right] \hat{U}_{\boldsymbol{n}}(\theta) &= \hat{U}^\dagger_{\boldsymbol{n}}(\theta)\hat{\boldsymbol{M}}\hat{U}_{\boldsymbol{n}}(\theta) \cdot (R_{\boldsymbol{n}}(\theta)\boldsymbol{B}) \\ &= \left(\hat{\boldsymbol{M}} + i\theta\sum_{i=1}^{3} n_i[\hat{L}_i, \hat{\boldsymbol{M}}]\right) \cdot (\boldsymbol{B} + \theta\boldsymbol{n} \times \boldsymbol{B}) + \mathcal{O}(\theta^2).\end{aligned} \tag{15}$$

Written out in components we obtain

$$\begin{aligned}&\sum_{j=1}^{3}\left(\hat{M}_j + i\theta\sum_{i=1}^{3} n_i[\hat{L}_i, \hat{M}_j]\right)\left(B_j + \theta\sum_{pq=1}^{3}\epsilon_{jpq}n_pB_q\right) + \mathcal{O}(\theta^2) \\ &= \sum_{j=1}^{3}\hat{M}_jB_j + i\theta\sum_{i,j=1}^{3} n_i[\hat{L}_i, \hat{M}_j]B_j + \theta\sum_{i,j,k=1}^{3}\epsilon_{kij}n_iB_j\hat{M}_k + \mathcal{O}(\theta^2).\end{aligned} \tag{16}$$

Since the first term is the Hamiltonian before the rotation the terms proportional to $\theta$ must cancel to zero for arbitrary choices of the $n_j$ and $B_i$, yielding the condition

$$i[\hat{L}_i, \hat{M}_j] = -\sum_{k=1}^{3}\epsilon_{ijk}\hat{M}_k, \tag{17}$$

or, equivalently,

$$[\hat{L}_i, \hat{M}_j] = i\sum_{k=1}^{3}\epsilon_{ijk}\hat{M}_k. \tag{18}$$

Hence, $\hat{\boldsymbol{M}}$ is a vector operator.

2. The existence of this eigenbasis follows directly from the fact that $\hat{H}_0$ commutes with $\hat{\boldsymbol{L}}$. We can use the additional quantum number $n$ to lift degeneracies of the subspaces of $\hat{\boldsymbol{L}}$. The Wigner-Eckart theorem tells us that the eigenvalues will not depend on $m_l$: We have seen that $\hat{H}_0$, in the absence of a magnetic field, is a scalar operator ($k = q = 0$). The Wigner-Eckart theorem states that

$$\langle n', l', m_l'|\hat{H}_0|n, l, m_l\rangle = \alpha_{n',n,l',l}\langle l', m_l'|0, 0; l, m_l\rangle, \tag{19}$$

where the Clebsch-Gordan coefficients $\langle l', m_l'|0, 0; l, m_l\rangle$ are zero unless $l' = l$ and $m_l = m_l'$. By picking $n$ as the quantum number that labels energy eigenstates, $\hat{H}_0$ becomes diagonal in the basis $|n, l, m_l\rangle$. The proportionality factors $\alpha$ will only depend on $n$ and $l$ (but not on $m_l$) and correspond to the diagonal elements, i.e., the eigenenergies.

3. We know that vector operators are, in each subspace $|n, l\rangle$, proportional to the angular momentum operator and we can write

$$\hat{\boldsymbol{M}} = \frac{\langle\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{L}}\rangle_{n,l}}{\langle\hat{\boldsymbol{L}}^2\rangle_{n,l}}\hat{\boldsymbol{L}}. \tag{20}$$