## Problem 1 Tensor product decomposition

You saw that for each $l$ there exist an (a priori complex) irreducible representation $\mathcal{E}_l$ of $SU(2)$ of dimension $2l+1$, spanned by vectors $|l,m\rangle$, $m \in \{-l,-l+1,...,l-1,l\}$. It is usually called spin-$l$-representation. We can identify $\mathcal{E}_l \cong \mathbb{C}^{2l+1}$.

1. Using Clebsch-Gordan theory, decompose $\mathcal{E}_1 \otimes \mathcal{E}_1$ and $\mathcal{E}_{1/2} \otimes \mathcal{E}_{1/2}$ into irreducible representations of $SU(2)$.
2. We would like to understand how this decomposition exactly works. Consider the defining (3-dim.) representation of $SO(3)$ on $\mathcal{E}_1$, $\rho(R)v = Rv$ where $R \in SO(3)$ and $v \in \mathbb{R}^3$ (in odd dimensions, the (a priori complex) spin-$k$ representations can be made real). Think of the space $\mathcal{E}_1 \otimes \mathcal{E}_1$ as $3\times 3$ matrices that can be decomposed into symmetric and anti-symmetric subspaces. Show that the tensor product representation of the defining representation of $SO(3)$, $\rho\otimes\rho$, leaves them invariant and identify the remaining 1-dim. subspace that can be extracted from the symmetric subspace, such that all subspaces are irreducible under $SO_3$.
3. Consider the defining (2-dim.) representation of $SU(2)$ on $\mathcal{E}_{1/2}$ and decompose $\mathcal{E}_{1/2} \otimes \mathcal{E}_{1/2}$ into irreducible subspaces as above.

## Problem 2 Magnetic moment and gyromagnetic factor

Consider a quantum system $\mathcal{S}$ described by a Hilbert space $\mathcal{H}$. We call $\hat{\boldsymbol{L}}$ its total angular momentum (including spin). In the presence of a magnetic field $\boldsymbol{B}$, we write its Hamiltonian as $\hat{H}(\boldsymbol{B})$. For small fields, we expand $\hat{H}$ as

$$\hat{H}(\boldsymbol{B}) = \hat{H}_0 - \sum_\alpha \hat{M}_\alpha B_\alpha + \dots \tag{2}$$

where, by definition, $\hat{H}_0 = \hat{H}(0)$ and $\hat{M}_\alpha = -\partial_{B_\alpha} H(\boldsymbol{B}=0)$.

1. Assuming that the total system (system and field) is rotationally symmetric, show that $\hat{H}_0$ and $\hat{\boldsymbol{M}}$ are respectively scalar (rank 0) and vector (rank 1) tensor. A rank $k$ tensor is an is an element of the vector space of operators on Hilbert space that transforms in a spin-$k$ representation of $\mathfrak{su}(2)$ or $SU(2)$. *Hint:* Consider the effect of infinitesimal rotations on $\boldsymbol{B}$ and $\hat{H}$.
2. We consider first the case of a zero external magnetic field. Let us decompose the system Hilbert space into eigenstates of the angular momentum operator $\hat{L}_z$ as $|n,\ell,m_\ell\rangle = |n\rangle \otimes |\ell,m_\ell\rangle$, where $|l,m_l\rangle$ forms for each $l$ an irreducible $\mathfrak{su}(2)$ representation. Here $n$ labels additional degrees of freedom, that are not touched by rotations, and it is chosen such that $\hat{H}_0$ is diagonal when acting of $|n\rangle$. Show that the eigenstates of $\hat{H}_0$ are also of the form $|n,\ell,m_\ell\rangle$. Using Winger Eckart's theorem, show that the corresponding energies are of the form $E_{n,\ell}$, i.e. independent of $m_l$.
3. We now consider the case of a weak (albeit finite) magnetic field. Using the Wigner Eckart theorem, show that, at first order in perturbation, the energy spectrum can be interpreted by endowing the system with a magnetic moment $\hat{\boldsymbol{M}}_{n\ell} = \gamma_{n\ell}\hat{\boldsymbol{L}}$.