Problem 1 The Heisenberg group

1. Consider the set $\mathcal{H}$ of 3×3 matrices defined by

$$M(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c \in \mathbb{R}. \tag{1}$$

   a) Show that $\mathcal{H}$ is a Lie group under matrix multiplication.
   b) Give an explicit basis of the corresponding Lie algebra $\mathfrak{h}$.
   c) Show that $\mathfrak{h}$ is spanned by three operators (L_1, L_2, L_3) obeying the following commutation relations

$$[L_1, L_2] = L_3 \qquad [L_1, L_3] = [L_2, L_3] = 0 \tag{2}$$

   d) A Lie algebra endowed with this structure is known as an *Heisenberg* algebra. Show that in quantum mechanics the operators $(\hat{x}, \hat{p}, i\hbar)$ span a Heisenberg algebra. What about $(\hat{a}, \hat{a}^\dagger, 1)$, where $\hat{a}$ and $\hat{a}^\dagger$ are bosonic creation and annihilation operators?

2. Consider now a general Heisenberg algebra spanned by three operators (L_1, L_2, L_3) obeying the commutation relations (2).
   a) For any $\lambda \in \mathbb{C}$, calculate $e^{\lambda L_1} L_2 e^{-\lambda L_1}$. What relation do we recover in the case of the operators $\hat{x}$ and $\hat{p}$?

3. *Glauber's relation.* We want here to relate $e^{L_1}e^{L_2}$ and $e^{L_1+L_2}$. For this, let us introduce $F(\lambda)$ defined by

$$F(\lambda) = e^{\lambda L_1} e^{\lambda L_2} e^{-\lambda^2 L_3/2}. \tag{3}$$

   a) Using the result of 2, show that $F'(\lambda) = (L_1 + L_2)F(\lambda)$.
   b) Solve this differential equation and deduce Glauber's relation

$$e^{L_1}e^{L_2} = e^{L_1+L_2}e^{[L_1,L_2]/2}. \tag{4}$$

Solution to Problem 1:

1. a) We first check the group property:

$$M(a,b,c)\cdot M(a',b',c') = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a'+a & c'+c+ab' \\ 0 & 1 & b'+b \\ 0 & 0 & 1 \end{pmatrix}$$
$$= M(a+a', b+b', c+c'+ab'). \tag{5}$$

We see that $M(0,0,0)$ acts as neutral element and

$$M(a,b,c)^{-1} = M(-a,-b,ab-c). \tag{6}$$

Furthermore, the elements are smoothly parametrized by three real parameters, hence, $(\mathcal{H}, \cdot)$ is a Lie group.

b) We can write $M(a,0,0)$ as $M(a,0,0) = \mathrm{Id} + aL_1$, where

$$L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Hence, L_1 is in the tangent space of $\mathcal{H}$ and part of the Lie algebra. It is easy to see that the matrices L_i are nilpotent and thus $\mathrm{Id} + aL_1 = e^{aL_1}$. It would suffice to expand the exponential for small a but here it holds generally. Similarly, we identify the tangent vectors

$$L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

c) Direct verification confirms the commutation relations (2).

d) We know that $[\hat{x}, \hat{p}] = i\hbar$ and $[\hat{a}, \hat{a}^\dagger] = 1$. Since the commutator is in both cases proportional to the identity operator, the operators commute with their commutator and yield a Heisenberg algebra.

2. Defining $f(\lambda) = e^{\lambda L_1} L_2 e^{-\lambda L_1}$, we find that

$$\begin{aligned} f'(\lambda) &= e^{\lambda L_1}[L_1, L_2]e^{-\lambda L_1} \\ &= L_3, \end{aligned} \tag{9}$$

where we used that L_1 and L_2 commute with L_3; see Eq. (2). Integrating this equation yields $f(\lambda) - f(0) = \lambda L_3$, where $f(0) = L_2$ and therefore

$$f(\lambda) = L_2 + \lambda L_3. \tag{10}$$

In the case of $\hat{x}$ and $\hat{p}$, we find that translations of $\hat{x}$ are generated by $\hat{p}$ and vice-versa. Choosing, e.g., $\lambda = iq/\hbar$, we find

$$e^{iq\hat{p}/\hbar}\hat{x}e^{-iq\hat{p}/\hbar} = \hat{x} - q. \tag{11}$$

We can interpret the translations with complex exponent as elements of the real Lie group that is generated when the generators are multiplied by the imaginary unit i.

3. We find

a)

$$F'(\lambda) = L_1 e^{\lambda L_1} e^{\lambda L_2} e^{-\lambda^2 L_3/2} + e^{\lambda L_1} L_2 e^{\lambda L_2} e^{-\lambda^2 L_3/2} - \lambda e^{\lambda L_1} e^{\lambda L_2} L_3 e^{-\lambda^2 L_3/2}. \tag{12}$$

We rewrite Eq. (10) as $e^{\lambda L_1} L_2 = (L_2 + \lambda L_3)e^{\lambda L_1}$ and insert it above to obtain

$$\begin{aligned} F'(\lambda) = &L_1 e^{\lambda L_1} e^{\lambda L_2} e^{-\lambda^2 L_3/2} \\ &+ (L_2 + \lambda L_3) e^{\lambda L_1} e^{\lambda L_2} e^{-\lambda^2 L_3/2} \\ &- \lambda e^{\lambda L_1} e^{\lambda L_2} L_3 e^{-\lambda^2 L_3/2}. \end{aligned} \tag{13}$$

Moreover, L_3 commutes with both L_1 and L_2, leading to

$$F'(\lambda) = (L_1 + L_2)F(\lambda). \tag{14}$$

The initial condition is $F(0) = \mathrm{Id}$.

b) We obtain a differential equation of exponential type that is solved by the ansatz

$$F(\lambda) = e^{(L_1+L_2)\lambda}. \tag{15}$$

This implies

$$e^{(L_1+L_2)\lambda} = e^{\lambda L_1} e^{\lambda L_2} e^{-\lambda^2 L_3/2}. \tag{16}$$

Using $\lambda = 1$ and $L_3 = [L_1, L_2]$ yields Glauber's relation

$$e^{L_1} e^{L_2} = e^{L_1+L_2} e^{[L_1,L_2]/2}. \tag{17}$$

Problem 2 The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$

Consider a quantum system of n modes or particles with $2n$ phase space observables

$$\hat{\boldsymbol{r}} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n)^T. \tag{18}$$

These operators satisfy the canonical commutation relations

$$[\hat{x}_k, \hat{p}_l] = i\hbar\delta_{kl}. \tag{19}$$

1. Find the $2n \times 2n$ matrix J that describes the commutation relations in the compact form

$$[\hat{r}_i, \hat{r}_j] = i\hbar J_{ij}. \tag{20}$$

   Show that J defines a non-degenerate, skew-symmetric bilinear form on $\mathbb{R}^{2n}$.

2. Transformations of the phase-space observables are described by the action of real $2n \times 2n$ matrices S as

$$\hat{\boldsymbol{r}}' = S\hat{\boldsymbol{r}}. \tag{21}$$

   Canonical transformations are those that conserve the commutation relations. Show that canonical transformations form a Lie group. This group is known as the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. Show that symplectic matrices can be written as

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{22}$$

   where A, B, C, D are real $n \times n$ matrices satisfying

$$\begin{aligned} AB^T - BA^T &= \mathbf{0}_n \\ CD^T - DC^T &= \mathbf{0}_n \\ AD^T - BC^T &= \mathbf{1}_n. \end{aligned} \tag{23}$$

3. Consider a Hamiltonian of second order in the phase space observables, written as
$$\hat{H} = \frac{1}{2}\hat{\boldsymbol{r}}^T H \hat{\boldsymbol{r}} = \frac{1}{2}\sum_{kl} H_{kl}\hat{r}_k\hat{r}_l, \tag{24}$$
where the H_{kl} form a real symmetric $2n \times 2n$ matrix H of coefficients. The evolution of phase space observables is described in the Heisenberg picture by
$$\hat{\boldsymbol{r}}(t) = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{r}}e^{-i\hat{H}t/\hbar}. \tag{25}$$
Show that this transformation is canonical and find the symplectic matrix S_H that satisfies
$$\hat{\boldsymbol{r}}(t) = S_H\hat{\boldsymbol{r}}. \tag{26}$$

4. Find the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$. Show that $JH \in \mathfrak{sp}_{2n}(\mathbb{R})$.
5. Find a block decomposition of $L \in \mathfrak{sp}_{2n}(\mathbb{R})$, in analogy to (22). What is the dimension of $\mathrm{Sp}_{2n}(\mathbb{R})$?

Solution to Problem 2:

1. We can easily verify that the matrix J is given by
$$J = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \tag{27}$$
where $\mathbf{0}_n$ and $\mathbf{1}_n$ are the n-dimensional zero and identity matrices, respectively.
This $2n \times 2n$ matrix defines a bilinear form on $\mathbb{R}^{2n}$ as
$$\omega(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}^T J\boldsymbol{w}. \tag{28}$$
Recall that non-degenerate means $\omega(\boldsymbol{v}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w}$ implies $\boldsymbol{v} = \mathbf{0}$. This is true if and only if $\det(J) \neq 0$. We see that $J^T J = \mathbf{1}_{2n}$. It follows that
$$1 = \det(JJ^T) = \det(J)^2, \tag{29}$$
hence, J (and ω) is non-degenerate. We can also see directly that $\det(J) = 1$ by bringing it to the form $\mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n)$ with n transpositions of rows. The transpositions give a factor $(-1)^n$ and the determinant of the diagonal matrix is also $(-1)^n$, leading to $\det(J) = (-1)^{2n} = 1$.
Moreover, we find that $J^T = -J$ and
$$\omega(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{w}^T J\boldsymbol{v} = \boldsymbol{v}^T J^T\boldsymbol{w} = -\boldsymbol{v}^T J\boldsymbol{w} = -\omega(\boldsymbol{v}, \boldsymbol{w}), \tag{30}$$
so ω is skew-symmetric (since J is). A vector space equipped with a non-degenerate, skew-symmetric bilinear form is called a symplectic space.

2. We check the commutation relations after the transformation S has been applied:

$$\begin{aligned}
[\hat{r}'_k, \hat{r}'_l] &= [\sum_p S_{kp}\hat{r}_p, \sum_q S_{lq}\hat{r}_q] \\
&= \sum_{pq} S_{kp}S_{lq}[\hat{r}_p, \hat{r}_q] \\
&= i\hbar \sum_{pq} S_{kp}S_{lq}J_{pq} \\
&= i\hbar (SJS^T)_{kl}.
\end{aligned} \tag{31}$$

The commutation relations are preserved by S if and only if

$$SJS^T = J. \tag{32}$$

Canonical transformations thus belong to the set

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \{S \in \mathcal{M}_{2n}(\mathbb{R}) \mid SJS^T = J\}, \tag{33}$$

also known as symplectic matrices.

Let us show that $\mathrm{Sp}_{2n}(\mathbb{R})$ indeed forms a group. It follows immediately that the identity matrix $\mathbf{1}_{2n} \in \mathrm{Sp}_{2n}(\mathbb{R})$ and that the product of two symplectic matrices is again symplectic. It only remains to be shown that each symplectic matrix is invertible and that the inverse is again symplectic. To see this, we multiply (32) from the right by J^T, leading to

$$SJS^TJ^T = \mathbf{1}_{2n}. \tag{34}$$

It follows that

$$S^{-1} = JS^TJ^T, \tag{35}$$

hence, every symplectic matrix is invertible. To see that S^{-1} is also symplectic, consider

$$\underbrace{S^{-1}}_{JS^TJ^T} J \underbrace{(S^{-1})^T}_{(S^T)^{-1}} = JS^T \underbrace{J^TJ}_{\mathbf{1}_{2n}} (S^T)^{-1} = J. \tag{36}$$

Hence, $\mathrm{Sp}_{2n}(\mathbb{R})$ is a matrix group.

By writing S as in Eq. (22), we obtain

$$\begin{aligned}
SJS^T &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} B^T & D^T \\ -A^T & -C^T \end{pmatrix} \\
&= \begin{pmatrix} AB^T - BA^T & AD^T - BC^T \\ CB^T - DA^T & CD^T - DC^T \end{pmatrix}.
\end{aligned} \tag{37}$$

The symplectic condition (32) now implies (23). These constraints allow for a total of $2n^2 + n$ real parameters that continuously parametrize this group (as we will see explicitly below).

Since matrix multiplication and inverse are smooth functions (they can be expressed in terms of polynomials of the matrix elements), $\mathrm{Sp}_{2n}(\mathbb{R})$ is a Lie group.

3. Let us first note that the commutation relations are conserved by any unitary transformation of the phase-space observables: For $\hat{r}'_j = \hat{U}^\dagger \hat{r}_j \hat{U}$, we obtain

$$\begin{aligned}
[\hat{r}'_j, \hat{r}'_k] &= [\hat{U}^\dagger \hat{r}_j \hat{U}, \hat{U}^\dagger \hat{r}_k \hat{U}] \\
&= \hat{U}^\dagger [\hat{r}_j, \hat{r}_k] \hat{U} \\
&= [\hat{r}_j, \hat{r}_k],
\end{aligned} \tag{38}$$

since $[\hat{r}_j, \hat{r}_k]$ is proportional to the identity operator for all j and k.

Now, we consider the Heisenberg equation of motion for $\hat{r}_j(t)$ and we obtain (omitting the time argument)[1]

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{r}_j &= \frac{i}{\hbar}[\hat{H}, \hat{r}_j] \\
&= \frac{i}{2\hbar}\sum_{kl} H_{kl}[\hat{r}_k\hat{r}_l, \hat{r}_j] \\
&= \frac{i}{2\hbar}\sum_{kl} H_{kl}(\hat{r}_k\hat{r}_l\hat{r}_j - \hat{r}_j\hat{r}_k\hat{r}_l \underbrace{-\hat{r}_k\hat{r}_j\hat{r}_l + \hat{r}_k\hat{r}_j\hat{r}_l}_{=0}) \\
&= \frac{i}{2\hbar}\sum_{kl} H_{kl}(\hat{r}_k[\hat{r}_l, \hat{r}_j] + [\hat{r}_k, \hat{r}_j]\hat{r}_l) \\
&= -\frac{1}{2}\sum_{kl} H_{kl}(J_{lj}\hat{r}_k + J_{kj}\hat{r}_l) \\
&= \sum_{kl} J_{jk}H_{kl}\hat{r}_l,
\end{aligned} \tag{39}$$

where we used that $[\hat{r}_l(t), \hat{r}_j(t)] = J_{lj}$ for all t because of (38). In the last step we used that H is symmetric and J is skew-symmetric. We obtain

$$\frac{\partial}{\partial t}\hat{\boldsymbol{r}} = JH\hat{\boldsymbol{r}}. \tag{40}$$

This equation is solved by $\hat{\boldsymbol{r}}(t) = S_H\hat{\boldsymbol{r}}(0)$ with $\hat{\boldsymbol{r}}(0) = \hat{\boldsymbol{r}}$ and

$$S_H = e^{JHt}. \tag{41}$$

From the previous exercise we conclude that this must indeed correspond to a symplectic matrix since we have a linear combination of the form (21) that conserves the commutator relations.

4. To find the Lie algebra, we consider a smooth path in the Lie group, parametrized by t and starting from the identity. We then expand

$$S = \mathbf{1}_{2n} + tL + \mathcal{O}(t^2), \tag{42}$$

where $L \in \mathfrak{sp}_{2n}(\mathbb{R})$. We obtain

$$SJS^T = J + t(LJ + JL^T) + \mathcal{O}(t^2), \tag{43}$$

[1] Note that $e^{i\hat{H}t}\hat{H}e^{-i\hat{H}t} = \hat{H}$ implies $\sum_{kl} H_{kl}\hat{r}_k\hat{r}_l = \sum_{kl} H_{kl}\hat{r}_k(t)\hat{r}_l(t)$ for all t.

and thus the condition (32) requires that $LJ + JL^T = \mathbf{0}_{2n}$. We thus find

$$\mathfrak{sp}_{2n}(\mathbb{R}) = \{L \in \mathcal{M}_{2n}(\mathbb{R}) \mid LJ + JL^T = \mathbf{0}_{2n}\}. \tag{44}$$

One can check explicitly that the exponential of such a matrix yields a symplectic matrix, but it is sufficient to verify this in the vicinity of the identity as we did above.

We saw in the previous exercise that e^{JHt} yields a symplectic matrix, so $JH \in \mathfrak{sp}_{2n}(\mathbb{R})$. We can verify the defining property explicitly: Inserting $L = JH$ and using the symmetry of H yields $JHJ + JHJ^T = JHJ - JHJ = 0$.

5. By writing

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{45}$$

we obtain

$$\begin{aligned} LJ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \\ &= \begin{pmatrix} -B & A \\ -D & C \end{pmatrix}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} JL^T &= \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \\ &= \begin{pmatrix} B^T & D^T \\ -A^T & -C^T \end{pmatrix}. \end{aligned} \tag{47}$$

The condition $LJ + JL^T = \mathbf{0}_{2n}$ then implies

$$\begin{aligned} B &= B^T \\ C &= C^T \\ D &= -A^T. \end{aligned} \tag{48}$$

We now use this to determine the dimension of the tangent space, i.e., the Lie algebra (which equals the dimension of the manifold, i.e., the Lie group). Note that we can independently choose two real $n \times n$ symmetric matrices B and C (with $n(n+1)/2$ free parameters each), and an arbitrary matrix A (with n^2 free parameters). This yields a total of

$$\dim \mathfrak{sp}_{2n}(\mathbb{R}) = \dim \mathrm{Sp}_{2n}(\mathbb{R}) = 2n^2 + n \tag{49}$$

parameters.