Problem 1 The Heisenberg group

1. Consider the set $\mathcal{H}$ of 3×3 matrices defined by

$$M(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c \in \mathbb{R}. \tag{1}$$

   a) Show that $\mathcal{H}$ is a Lie group under matrix multiplication.

   b) Give an explicit basis of the corresponding Lie algebra $\mathfrak{h}$.

   c) Show that $\mathfrak{h}$ is spanned by three operators (L_1, L_2, L_3) obeying the following commutation relations

$$[L_1, L_2] = L_3 \qquad [L_1, L_3] = [L_2, L_3] = 0 \tag{2}$$

   d) A Lie algebra endowed with this structure is known as an *Heisenberg* algebra. Show that in quantum mechanics the operators $(\hat{x}, \hat{p}, i\hbar)$ span a Heisenberg algebra. What about $(\hat{a}, \hat{a}^\dagger, 1)$, where $\hat{a}$ and $\hat{a}^\dagger$ are bosonic creation and annihilation operators?

2. Consider now a general Heisenberg algebra spanned by three operators (L_1, L_2, L_3) obeying the commutation relations (2).

   a) For any $\lambda \in \mathbb{C}$, calculate $e^{\lambda L_1} L_2 e^{-\lambda L_1}$. What relation do we recover in the case of the operators $\hat{x}$ and $\hat{p}$?

3. *Glauber's relation.* We want here to relate $e^{L_1} e^{L_2}$ and $e^{L_1 + L_2}$. For this, let us introduce $F(\lambda)$ defined by

$$F(\lambda) = e^{\lambda L_1} e^{\lambda L_2} e^{-\lambda^2 L_3/2}. \tag{3}$$

   a) Using the result of 2, show that $F'(\lambda) = (L_1 + L_2) F(\lambda)$.

   b) Solve this differential equation and deduce Glauber's relation

$$e^{L_1} e^{L_2} = e^{L_1 + L_2} e^{[L_1, L_2]/2}. \tag{4}$$

Problem 2 The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$

Consider a quantum system of n modes or particles with $2n$ phase space observables

$$\hat{\boldsymbol{r}} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n)^T. \tag{18}$$

These operators satisfy the canonical commutation relations

$$[\hat{x}_k, \hat{p}_l] = i\hbar \delta_{kl}. \tag{19}$$

1. Find the $2n \times 2n$ matrix J that describes the commutation relations in the compact form

$$[\hat{r}_i, \hat{r}_j] = i\hbar J_{ij}. \tag{20}$$

   Show that J defines a non-degenerate, skew-symmetric bilinear form on $\mathbb{R}^{2n}$.

2. Transformations of the phase-space observables are described by the action of real $2n \times 2n$ matrices S as

$$\hat{\boldsymbol{r}}' = S\hat{\boldsymbol{r}}. \tag{21}$$

Canonical transformations are those that conserve the commutation relations. Show that canonical transformations form a Lie group. This group is known as the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. Show that symplectic matrices can be written as

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{22}$$

where A, B, C, D are real $n \times n$ matrices satisfying

$$\begin{aligned} AB^T - BA^T &= \mathbf{0}_n \\ CD^T - DC^T &= \mathbf{0}_n \\ AD^T - BC^T &= \mathbf{1}_n. \end{aligned} \tag{23}$$

3. Consider a Hamiltonian of second order in the phase space observables, written as

$$\hat{H} = \frac{1}{2}\hat{\boldsymbol{r}}^T H \hat{\boldsymbol{r}} = \frac{1}{2}\sum_{kl} H_{kl}\hat{r}_k\hat{r}_l, \tag{24}$$

where the H_{kl} form a real symmetric $2n\times 2n$ matrix H of coefficients. The evolution of phase space observables is described in the Heisenberg picture by

$$\hat{\boldsymbol{r}}(t) = e^{i\hat{H}t/\hbar}\hat{\boldsymbol{r}}e^{-i\hat{H}t/\hbar}. \tag{25}$$

Show that this transformation is canonical and find the symplectic matrix S_H that satisfies

$$\hat{\boldsymbol{r}}(t) = S_H\hat{\boldsymbol{r}}. \tag{26}$$

4. Find the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$. Show that $JH \in \mathfrak{sp}_{2n}(\mathbb{R})$.

5. Find a block decomposition of $L \in \mathfrak{sp}_{2n}(\mathbb{R})$, in analogy to (22). What is the dimension of $\mathrm{Sp}_{2n}(\mathbb{R})$?