Problem 1 Path-connectedness of $SO(n)$

Show that $SO(n)$ is path connected using induction over n. *Hint:* It is enough to show that there is a path from each element to the identity.

Problem 2 Time reversal in quantum mechanics

Operations $f(x)$ that conserve the modulus $|\langle f(x)|f(y)\rangle| = |\langle x|y\rangle|$, $\forall (x,y) \in \mathcal{H}^2$, where $\mathcal{H}$ is a Hilbert space with Hermitian product $\langle\cdot|\cdot\rangle$, can only be of a certain kind: Wigner's theorem states that f can be recast as $f(x) = e^{i\theta(x)}U(x)$, where θ is real-valued and $U : \mathcal{H} \to \mathcal{H}$ obeys the following properties

$$\forall (x,y) \in \mathcal{H}^2 \qquad U(x+y) = U(x) + U(y) \tag{4}$$
$$\forall (\lambda, x) \in \mathbb{C} \times \mathcal{H} \qquad U(\lambda x) = \kappa(\lambda) U(x) \tag{5}$$
$$\forall (x,y) \in \mathcal{H}^2 \qquad \langle U(x)|U(y)\rangle = \kappa(\langle x|y\rangle) \tag{6}$$

where $\kappa(\lambda) = \lambda$ (unitary operators) or $\kappa(\lambda) = \bar{\lambda}$ (anti-unitary operators).
In this exercise, we focus on time reversal symmetry $T : \psi(\boldsymbol{r}) \mapsto \overline{\psi(\boldsymbol{r})}$.

1. Consider the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{\boldsymbol{r}})$. Find an interpretation of T by considering its action on solutions of Schrödinger's equation $i\hbar\partial_t\psi = \hat{H}\psi$.
2. Show that T is an anti-unitary operator.
3. Consider a 1D problem and assume V has compact support.
   (a) Prove the existence of the two families of solutions:
   $$\psi_+(x) = \begin{cases} e^{ikx} + r_+ e^{-ikx} & x \to -\infty \\ t_+ e^{ikx} & x \to +\infty \end{cases}, \quad \psi_-(x) = \begin{cases} t_- e^{-ikx} & x \to -\infty \\ e^{-ikx} + r_- e^{ikx} & x \to +\infty \end{cases}. \tag{7}$$
   (b) Assuming the solution has time reversal symmetry, find the relations between $r_\pm$ and $t_\pm$.
4. Is the interpretation of T as a solution of the Schrödinger equation after time reversal still valid if we add an external magnetic field, $\hat{H} = \frac{1}{2m}\left(-i\hbar\nabla + \frac{q}{c}\mathbf{A}\right)^2 + V(\hat{\boldsymbol{r}})$?

Problem 3 (Bonus) Crystal symmetries and Pockels effect

In dielectric media, the displacement field $\mathbf{D} \in \mathbb{R}^3$ is related to the electric field $\mathbf{E} \in \mathbb{R}^3$ by the relation $D^i = \epsilon^i_j E^j$, where the ϵ^i_j are the components of the permittivity tensor (we make use of the sum convention). The Pockels effect describes a change of the permittivity tensor due to an electric field as $\epsilon^i_j = L^i_{jk} E^k$, where L^i_{jk} are the components of a rank-3 tensor.

1. Identify the most general form of the permittivity tensor in a crystal with C_4 symmetry.
2. Show that the Pockels effect cannot be observed in crystals with inversion symmetry.