Problem 1 Representations of Abelian Groups

1. Using Schur's lemma, show that all irreducible representations of an Abelian group are one-dimensional.
2. Find all irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{C}$. Hint: Consider the action of the cyclic group on the complex unit circle. Convince yourself that the complex conjugate of any of the irreducible representations is a new valid irreducible representation. This is true in general.

Solution to Problem 1:

1. Let G be Abelian and $(r, \mathcal{E})$ an irreducible representation of G. Then $\forall g, g' \in G$, $r[g] \circ r[g'] = r[g'] \circ r[g]$. Schur's lemma says that if $r[g] \circ T = T \circ r[g]$ for all $g \in G$ and r and irreducible representation, then $\exists \lambda \in \mathbb{C}^*$ such that $T = \lambda \mathrm{Id}_{\mathcal{E}}$. For a given $g \in G$ we take $T = r[g]$ and find $r[g] = \lambda(g)\mathrm{Id}_{\mathcal{E}}$. The representation r therefore leaves every one-dimensional subspace of $\mathcal{E}$ invariant. Since it is irreducible, this is only possible if $\mathcal{E}$ is one-dimensional itself.
2. Since $\mathbb{Z}/n\mathbb{Z}$ is finite Abelian, all its irreducible representations must be one dimensional. According to $\sum_{i=1}^{c} d_i^2 = n$ with $d_i = 1$, we have $c = n$, so there must be n different representations. The cyclic groups can be pictured as n equidistant rotations on the unit circle, which are generated by $r[g] = e^{i2\pi/n}$. From this, we obtain $r[g^k] = r[g]^k = e^{i2\pi k/n}$. Alternatively, we may represent each group element in terms of larger steps (consisting of m elementary rotations), leading to
$$r_m[g^k] = e^{i2\pi mk/n}, \tag{1}$$
for $m = 1, \dots, n$. Let us now verify that the r_m are all inequivalent irreducible representations of $\mathbb{Z}/n\mathbb{Z}$. First we show that they are well defined: $r_m[g^{k+n}] = e^{i2\pi m(k+n)/n} = e^{i2\pi mk/n} e^{i2\pi m} = e^{i2\pi mk/n} = r_m[g^k]$. Furthermore, $r_m[g^k g^l] = r_m[g^{k+l}] = e^{i2\pi m(k+l)/n} = e^{i2\pi mk/n} e^{i2\pi ml/n} = r_m[g^k] r_m[g^l]$, so they are irreducible (since one dimensional) representations. Since $r_m[g] = e^{i2\pi m/n}$ is different for all m, the representations are all inequivalent. This means that we have found all irreducible representations. *Note:* Complex conjugation always provides a new, valid representation. Here, however, the complex conjugated ones already correspond to another representation with different m.

Problem 2 Representations

1. Representation on the dual space: Let r be representation of G on $\mathcal{E}$. Show that for a linear function $f : \mathcal{E} \to \mathbb{C}$
$$[r^*[g]]f(v) = f\left(r[g]^{-1} v\right) \tag{2}$$
is a representation of f.
2. Show that irreducible representations r over a $\mathbb{C}$-vector space $\mathcal{E}$ must have the property that $r[g] = \lambda \mathrm{Id}_{\mathcal{E}}$ with $\lambda \in \mathbb{C}$ for all g from the center $Z(G)$ of the group G.
3. Show that two one-dimensional irreducible representations are equivalent if and only if they are the same.

Solution to Problem 2:

1. We have that $r[g]r[g'] = r[gg'] = r[g'g] = r[g']r[g]$ for all $g' \in Z(G)$ and $g \in G$. Schur's lemma says that if $r[g]T = Tr[g]$ for all $g \in G$ then $T = \lambda \mathrm{Id}_{\mathcal{E}}$, which proves the statement.

2. Two one-dimensional irreducible representations r_1, r_2, with $r_i : G \to \mathbb{K}$ are equivalent if there exists an invertible intertwining operator $T : \mathbb{K} \to \mathbb{K}$, i.e., $Tr_1 = r_2 T$ where T is linear. The only invertible linear operations are $T(x) = \lambda \cdot x$ where $\lambda \in \mathbb{K}$ and $\lambda \neq 0$. In the definition of the intertwining operator, the scalar cancels on both sides and we obtain $r_1 = r_2$.

Problem 3 Character table of O and S_4

The goal of this exercise is to construct the character table of O and S_4. We have seen that the groups are isomorphic, so they have the same character table. Start out by identifying the dimensions of the table and fill out elements as you proceed.

1. Fill in the characters of the trivial irreducible representation. What are the dimensions of the remaining irreducible representations?

2. Find a representation that describes the permutation of the standard basis in $\mathbb{C}^4$. Take the determinant of each matrix and show that it provides an irreducible representation. What is the interpretation of this representation?

3. Calculate the characters of the $4D$ representation. Identify the invariant $1D$ and $3D$ subspaces. What is the representation in the $1D$ invariant subspace? Use this to derive the characters of an irreducible $3D$ representation r_{std}, called the standard representation.

4. We can construct a new representation by using the tensor product of two representations. Find the characters of the tensor products of all irreducible representations that you found so far. Are they irreducible?

5. Complete the character table using Schur orthogonality.

6. Decompose $r_{\mathrm{std}} \otimes r_{\mathrm{std}}$ into irreducible representations of S_4.

Solution to Problem 3:

We have seen that there are five conjugacy classes, so the character table will be 5×5.

1. The trivial representation has 1 in all classes (fill in first row). The group has order 24. Labeling with $\mathbf{d} = (d_5, d_4, d_3, d_2, 1)$ the dimensions of the 5 irreps with $d_5 \geq d_4 \geq d_3 \geq d_2 \geq 1$, we have $24 = d_5^2 + d_4^2 + d_3^2 + d_2^2 + 1$. Evidently, $d_5 \leq 4$. Let us assume $d_5 = 4$. Since $4^2 + 3^2 + \cdots = 16 + 9 + \cdots > 25$, we have $d_4 \leq 2$. For $\mathbf{d} = (4, 2, 2, 1, 1)$ we obtain 26 (too much) but for $\mathbf{d} = (4, 2, 1, 1, 1)$ we have 23 (not enough), so there is no solution with $d_5 = 4$. Taking $d_5 = 3$, we cannot have more than two irreps of dimension 3 since $3 * 3^2 = 27 > 24$. For $\mathbf{d} = (3, 3, 2, 1, 1)$ we get 24, which is the unique solution. We fill that in the column for the identity.

2. We take the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_4\}$ in $\mathbb{C}^4$. The representations of the elements

in each class are written as

$$r[e] = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad r[(12)] = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad r[(12)(34)] = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix},$$
$$r[(123)] = \begin{pmatrix} 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ & & & 1 \end{pmatrix}, \quad r[(1234)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

The determinants of block diagonal matrices can be determined using

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\det A)(\det B). \tag{4}$$

Furthermore, we can use

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = a \det \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \det \begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} + c \det \begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d \det \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}. \tag{5}$$

This yields

$$\det r[e] = \det r[(123)] = \det r[(12)(34)] = 1$$

and

$$\det r[(12)] = \det r[(1234)] = -1, \tag{6}$$

which corresponds to the sign representation: $(-1)^m$, where m is the number of 2-cycles needed to represent the permutation. The number m is unique modulo 2 and therefore the sign is unique. Hence, the product of two odd and two even permutations is even, and the product of an even and an odd permutation is odd. This is reflected by the sign representation since $\det AB = \det A \det B$.

3. The matrices (3) permute the entries of any vector $\mathbf{v} \in \mathbb{C}^4$. This means that vectors with identical entries everywhere will be invariant and $\mathrm{Span}_{\mathbb{C}}\{(1,1,1,1)\}$ is an invariant $1D$ subspace. On this subspace the action of the permutation is described by the trivial representation. The remaining $3D$ subspace is given by the vectors $\mathbf{v} = (v_1, v_2, v_3, v_4)$ with $v_1 + v_2 + v_3 + v_4 = 0$. Let us now determine the characters of the representation, i.e., the traces of the matrices (3). We obtain

	e	(12)	(12)(34)	(123)	(1234)
χ_{4D}	4	2	0	1	0

. (7)

Let us now determine the "modulus" of this character. We obtain

$$\langle \chi_{4D} | \chi_{4D} \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = \frac{1}{24} \left(1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2 \right) = \frac{48}{24} = 2. \tag{8}$$

We know that the modulus in general yields $\langle \chi_{4D} | \chi_{4D} \rangle = \sum_i m_i^2$, where m_i is the number of times the irreducible representation i appears. The only possibility to obtain 2 is to have two distinct irreducible representations appear once. We have already identified the trivial representation, but we can also verify explicitly that $\langle \chi_{4D} | \chi_{\text{trv}} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2 \chi_{\text{trv}}(g)^* = \frac{1}{24} (4 + 6 \cdot 2 + 8 \cdot 1) = 1$. Our representation must be of the form $r = r_{\text{trv}} \oplus r_{\text{std}}$ where r_{std} is some irreducible $3D$ representation. For the character this implies that

$$\chi = \chi_{\text{trv}} + \chi_{\text{std}}, \tag{9}$$

and we obtain by subtracting the trivial character on both sides that

	e	(12)	(12)(34)	(123)	(1234)
χ_{std}	3	1	-1	0	-1

. (10)

must be the character for an irreducible $3D$ representation. This is called the standard representation and we add it to the table. It can be interpreted as the permutation of the vertices of a tetrahedron in $\mathbb{R}^3$.

4. Under tensor multiplication we obtain new characters as the products of old characters. The only non-trivial example that we can construct so far is

	e	(12)	(12)(34)	(123)	(1234)
$\chi_{\text{std}} \otimes \chi_{\text{sgn}}$	3	-1	-1	0	1

. (11)

We may verify by calculating its modulus $\frac{1}{24}(3 + 6 + 3 + 6) = 1$ that it is indeed another irreducible representation.

5. Finally, we need to find another $2D$ representation. From orthogonality with all four representations, we obtain the conditions

$$2 + 6\chi_2[(12)] + 3\chi_2[(12)(34)] + 8\chi_2[(123)] + 6\chi_2[(1234)] = 0 \tag{12}$$
$$2 - 6\chi_2[(12)] + 3\chi_2[(12)(34)] + 8\chi_2[(123)] - 6\chi_2[(1234)] = 0 \tag{13}$$
$$6 + 6\chi_2[(12)] - 3\chi_2[(12)(34)] - 6\chi_2[(1234)] = 0 \tag{14}$$
$$6 - 6\chi_2[(12)] - 3\chi_2[(12)(34)] + 6\chi_2[(1234)] = 0. \tag{15}$$

Adding and subtracting (12) and (13) yields

$$4 + 6\chi_2[(12)(34)] + 16\chi_2[(123)] = 0 \tag{16}$$
$$\chi_2[(12)] = -\chi_2[(1234)], \tag{17}$$

respectively. Adding (14) to (15) we obtain

$$12 - 6\chi_2[(12)(34)] = 0. \tag{18}$$

We thus find from Eqs. (18) and (16) that

$$\chi_2[(12)(34)] = 2 \tag{19}$$
$$\chi_2[(123)] = -1. \tag{20}$$

Inserting this into Eq. (14), we obtain $\chi_2[(1234)] = 1 - \chi_2[(1234)] - \frac{1}{2}\chi_2[(12)(34)]$ and therefore

$$2\chi_2[(1234)] = 1 - \frac{1}{2}\chi_2[(12)(34)] = 0. \tag{21}$$

This completes the character of the two-dimensional representation. *Note:* This representation is related to the fact that we have a normal subgroup H composed by the class (12)(34) together with the identity (4 elements). This gives rise to a quotient group G/H of order 6, which can be shown to be non-Abelian, and therefore it must be isomorphic to D_3. The irreducible $2D$ representation we just found is the $2D$ irreducible representation of D_3. It describes the transformations of the three segments that we can build by connecting two midpoints of surfaces inside the tetrahedron. As the (12)(34) class is in the same coset as the identity it also has character 2. Three-cycles are rotations about 120°, which have trace -1. The two-cycles describe a reflection that in $2D$ have eigenvalues ± 1 and therefore character zero. The same holds for four-cycles since they are in the same coset as two-cycles.
The full character table reads:

C	e	(12)	(12)(34)	(123)	(1234)
λ	1+1+1+1	2+1+1	2+2	3+1	4
$\lvert C \rvert$	1	6	3	8	6
χ_{trv}	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1
χ_2	2	0	2	-1	0
χ_{std}	3	1	-1	0	-1
$\chi_{\text{std}} \otimes \chi_{\text{sgn}}$	3	-1	-1	0	1

(22)

6. We can decompose $r_{\text{std}} \otimes r_{\text{std}}$ into irreducibles using the character table. The character reads

	e	(12)	(12)(34)	(123)	(1234)
$\chi_{\text{std}} \otimes \chi_{\text{std}}$	9	1	1	0	1

. (23)

We determine the overlap with the different irreducible representations:

$$\langle \chi_{\text{std}} \otimes \chi_{\text{std}} | \chi_{\text{trv}} \rangle = \frac{1}{24}(9 + 6 + 3 + 6) = 1 \tag{24}$$

$$\langle \chi_{\text{std}} \otimes \chi_{\text{std}} | \chi_{\text{sgn}} \rangle = \frac{1}{24}(9 - 6 + 3 - 6) = 0 \tag{25}$$

$$\langle \chi_{\text{std}} \otimes \chi_{\text{std}} | \chi_2 \rangle = \frac{1}{24}(2 \cdot 9 + 3 \cdot 2) = 1 \tag{26}$$

$$\langle \chi_{\text{std}} \otimes \chi_{\text{std}} | \chi_{\text{std}} \rangle = \frac{1}{24}(27 + 6 - 3 - 6) = 1 \tag{27}$$

$$\langle \chi_{\text{std}} \otimes \chi_{\text{std}} | \chi_{\text{std}} \otimes \chi_{\text{sgn}} \rangle = \frac{1}{24}(27 - 6 - 3 + 6) = 1. \tag{28}$$

Hence, we have found that the 9-dimensional representation of $r_{\text{std}} \otimes r_{\text{std}}$ can be decomposed into a direct sum of four irreducibles with dimensions $1 + 2 + 3 + 3$, as

$$r_{\text{std}} \otimes r_{\text{std}} = r_{\text{trv}} \oplus r_2 \oplus r_{\text{std}} \oplus (r_{\text{std}} \otimes r_{\text{sgn}}). \tag{29}$$