Problem 1 Representations of Abelian Groups

1. Using Schur's lemma, show that all irreducible representations of an Abelian group are one-dimensional.
2. Find all irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{C}$. Hint: Consider the action of the cyclic group on the complex unit circle. Convince yourself that the complex conjugate of any of the irreducible representations is a new valid irreducible representation. This is true in general.

Problem 2 Representations

1. Representation on the dual space: Let r be representation of G on $\mathcal{E}$. Show that for a linear function $f : \mathcal{E} \to \mathbb{C}$
$$[r^*[g]]f(v) = f\left(r[g]^{-1}v\right) \tag{2}$$
is a representation of f.
2. Show that irreducible representations r over a $\mathbb{C}$-vector space $\mathcal{E}$ must have the property that $r[g] = \lambda \mathrm{Id}_{\mathcal{E}}$ with $\lambda \in \mathbb{C}$ for all g from the center $Z(G)$ of the group G.
3. Show that two one-dimensional irreducible representations are equivalent if and only if they are the same.

Problem 3 Character table of O and S_4

The goal of this exercise is to construct the character table of O and S_4. We have seen that the groups are isomorphic, so they have the same character table. Start out by identifying the dimensions of the table and fill out elements as you proceed.

1. Fill in the characters of the trivial irreducible representation. What are the dimensions of the remaining irreducible representations?
2. Find a representation that describes the permutation of the standard basis in $\mathbb{C}^4$. Take the determinant of each matrix and show that it provides an irreducible representation. What is the interpretation of this representation?
3. Calculate the characters of the $4D$ representation. Identify the invariant $1D$ and $3D$ subspaces. What is the representation in the $1D$ invariant subspace? Use this to derive the characters of an irreducible $3D$ representation r_{std}, called the standard representation.
4. We can construct a new representation by using the tensor product of two representations. Find the characters of the tensor products of all irreducible representations that you found so far. Are they irreducible?
5. Complete the character table using Schur orthogonality.
6. Decompose $r_{\text{std}} \otimes r_{\text{std}}$ into irreducible representations of S_4.