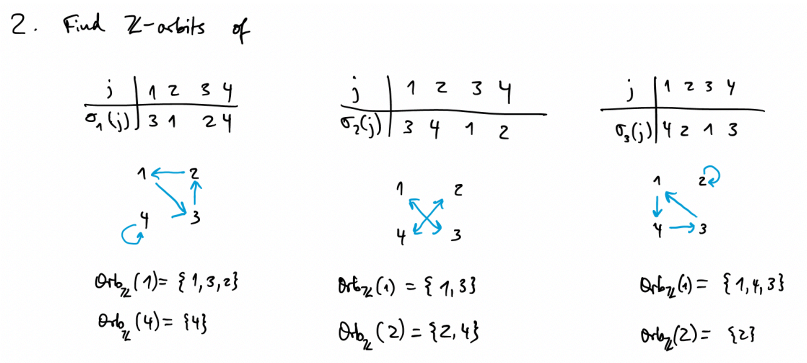
Problem 1 Symmetric group

1. Let G be a group and S_N the symmetric group, i.e., the group of permutations of $X = \{1, \ldots, N\}$. Show that any group homomorphism $\varphi : G \to S_N$ induces a group action on X by the action $g \cdot x = \varphi(g)(x)$. Here the action of an element $\sigma \in S_N$ on $x \in X$ is denoted by $\sigma(x)$.
2. Consider $\sigma \in S_4$ and the action of $G = \mathbb{Z}$ defined by the homomorphism
$$\begin{aligned} \varphi_\sigma : \mathbb{Z} &\to S_4 \\ j &\mapsto \sigma^j, \end{aligned} \tag{1}$$
where we interpret negative powers as powers of the inverse permutation. Find the orbits of this group action for the permutations σ of the following form.
$$\begin{array}{c|cccc} j & 1 & 2 & 3 & 4 \\ \hline \sigma_1(j) & 3 & 1 & 2 & 4 \end{array} \quad \begin{array}{c|cccc} j & 1 & 2 & 3 & 4 \\ \hline \sigma_2(j) & 3 & 4 & 1 & 2 \end{array} \quad \begin{array}{c|cccc} j & 1 & 2 & 3 & 4 \\ \hline \sigma_3(j) & 4 & 2 & 1 & 3 \end{array}. \tag{2}$$
3. One can denote the elements of S_N more efficiently by cycles. A cycle is defined by a $\mathbb{Z}$-orbit of a permutation with the elements written out in the order in which they occur. For example, the cycles $(142)(35)(6)$ corresponds to the following permutation in S_6: $1 \to 4, 4 \to 2, 2 \to 1, 3 \to 5, 5 \to 3, 6 \to 6$. Any permutation is uniquely characterized by its cycles. Cycles of length one (like (6) before) are sometimes omitted. Write out $\sigma_1, \sigma_2, \sigma_3$ and their inverses in terms of their cycles.
4. Which of the permutations σ_1, σ_2, σ_3 are in the same conjugacy class? Find $\tau \in S_4$ that relates two of the above elements by $\sigma' = \tau\sigma\tau^{-1}$ if they are in the same conjugacy class.
5. Show that for $\sigma, \tau \in S_N$ with $\sigma = (j_1, \ldots, j_k)$ a k-cycle, $\tau\sigma\tau^{-1} = (\tau(j_1), \ldots, \tau(j_k))$.
6. The cycle type of a permutation is given by the lengths of all of the disjoint cycles that appear in its decomposition. Prove that the conjugation classes of S_N are defined by the cycle type.
7. The cycle type can be visualized by so called Young diagram: Draw each k-cycle as a row of k squares, all cycles stacked on top of each other with larger cycles on the top, aligned on the left. For S_4, identify the diagrams describing σ_1, σ_2, and σ_3 and find the diagrams for the remaining conjugacy classes in S_4. How many elements does each conjugacy class have?
8. Do the same for S_5 and S_6 and find the general rule.
9. Find the minimal set of generators of S_N. The generators of a group are those elements that by repeated multiplication produce all other elements of the group. For example, the generators of the symmetry group of the regular triangle D_3 are rotations r by $2\pi/3$ and reflections s about any symmetry axis. One usually writes $D_3 = \langle r, s\rangle$.
10. Consider a cube centered at the origin. Assign four different labels to the eight vertices, putting the same label onto pairs of vertices at p and $-p$. Study the action on these labels for a rotation around a midpoint of a surface, a vortex, and a midpoint of an edge. Show that $S_4 \simeq O$.

Solution to Problem 1:

1. We check the axioms of a group action. First, since φ is a group homomorphism, we have $\varphi(e) = e$ is the identity permutation. Therefore $e \cdot (x) = \varphi(e)(x) = x$. Moreover, $g_2 \cdot g_1 \cdot x = g_2 \cdot \varphi(g_1)(x) = \varphi(g_2)\varphi(g_1)(x) = \varphi(g_2 g_1)(x) = (g_2 g_1) \cdot (x)$, so both axioms are satisfied. *Note:* Every group action can be uniquely represented in this way since $g \cdot x$ is an automorphism of X and can be identified with a permutation for every g.

2. We can obtain the orbits by writing up the elements in a circle and tracing their evolution under the group action:



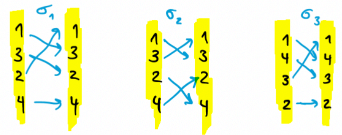
   Note: From the orbit decomposition, we have that the $\mathbb{Z}$-orbits of σ define a partition of X.

3. We have $\sigma_1 = (132)$, $\sigma_2 = (13)(24)$, and $\sigma_3 = (143)$. The inverses are given by reversing the order in each cycle, i.e., $\sigma_1^{-1} = (123), \sigma_2^{-1} = (13)(24) = \sigma_2$, and $\sigma_3^{-1} = (134)$.

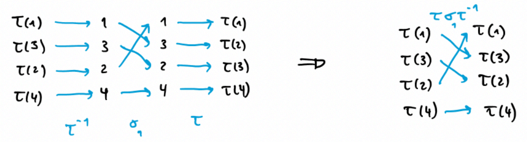
4. We represent the permutations by arrows between the elements (see below). We note that conjugation, i.e., $\tau\sigma\tau^{-1}$ can permute the elements of the domain and the codomain, but it cannot change the arrows in between them. We can write this down for a general permutation τ and see that for two permutations to be in the same conjugacy class, they need to have cycles of same length (this is not a formal proof of course). The permutation τ that relates σ_1 and σ_3 by conjugation can be explicitly constructed.

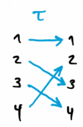
Conjugation: $\sigma \longrightarrow \tau\sigma\tau^{-1}$ $\Rightarrow$ permutation of domain and codomain of σ !

↳ changes the order within the yellow highlighted sets (in the same way for domain and codomain)

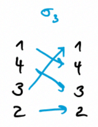


$\Rightarrow$ Conjugation can transform σ_1 into any





yields

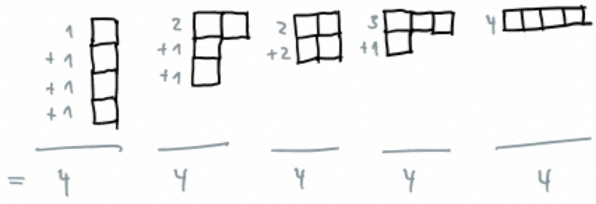


Not possible for σ_2.

5. We first consider what happens to an element $j \in X$ that is not part of the k-cycle. It must therefore be in a one-cycle of σ and we have $\sigma(j) = j$ and $\tau\sigma\tau^{-1}[\tau(j)] = \tau(j)$, so $\tau(j)$ is also in a one-cycle of $\tau\sigma\tau^{-1}$. We now focus on the elements j_l within the k-cycle. They satisfy $\sigma(j_l) = j_{l+1 \mod k}$ and $\tau\sigma\tau^{-1}[\tau(j_l)] = \tau[\sigma(j_l)] = \tau(j_{l+1 \mod k})$. This yields the result.

6. We have to prove that any two permutations are in the same conjugacy class if and only if they have the same cycle type. Since any permutation is decomposed into disjoint cycles (partition of X) it can be considered as the product of its cycles. Conjugation can then be performed cycle-wise, by inserting the identity $\tau\tau^{-1}$ in between two cycles. Each cycle will now transform according to the rule derived in the previous exercise, i.e., their cycle length will be conserved. This shows that conjugation cannot change the cycle type.
   Conversely, if two permutations σ, σ' have the same cycle type, we can write them as products of disjoint cycles $\sigma = s_1 \dots s_m$ and $\sigma' = s'_1 \dots s'_m$, where s_k and s'_k are cycles of the same length for all k. We can now define a permutation that maps all the elements of s_k onto these of s'_k, for all k. This is indeed a permutation of the full set since all cycles are disjoint. This yields $\tau\sigma\tau^{-1} = \sigma'$ and proves the statement.

7. See notes below:



$\sigma_1, \sigma_3 \in C_{(3,1)}$ $\qquad \sigma_2 \in C_{(2,2)}$

$C_{(1,1,1,1)} = \{e\}$.

$(12) \in C_{(2,1,1)}$

$(1342) \in C_{(4)}$

Number of elements in $C_{(1,1,1,1)} = 1$

In $C_{(2,1,1)}$: A two cycle is characterized by two different numbers, so there are $N(N-1)$ possibilities. However each two cycle can be witten equivalently in two different ways.

$$\Rightarrow \frac{N(N-1)}{2} = \frac{12}{2} = 6$$

In $C_{(3,1)}$: Three cycle has $N(N-1)(N-2)$ possibilities, but there are three different ways to write the same cycle.

$$\Rightarrow \frac{N(N-1)(N-2)}{3} = \frac{24}{3} = 8$$

In $C_{(2,2)}$: We have two two-cycles. Possibilities to fill them with numbers are $N(N-1)(N-2)(N-3)$ each one has two equivalent expressions. Furthermore, we can switch the two cycles.

$$\Rightarrow \frac{N(N-1)(N-2)(N-3)}{2 \cdot 2 \cdot 2} = \frac{24}{8} = 3$$

In $C_{(4)}$: There are again $N(N-1)(N-2)(N-3)$ possibilities, and four equivalent ways for each.

$\Rightarrow 6$.

8. A minimal set of generators of S_N is for example $\{(12), (12...N)\}$ (we could take any other transposition (ab) instead of (12) as long as $b - a$ and N don't have

any other common deviser other than 1). To see this, first notice that one can generate all transpositions between neighbours $S = \{(12), (23), (34), ...\}$ by repeated conjugation. E.g $(23) = (12...N)(12)(12...N)^{N-1}$. Starting from S, one can get any transposition, e.g. $(15) = (12)(23)(34)(45)(34)(23)(12)$. Finally one recovers an arbitrary cycle $(j_1, ..., j_k)$ from the corresponding transpositions as $(j_1, ..., j_k) = (j_1 j_2)(j_2 j_3)...(j_{k-1} j_k)$.

9. We see that each face has indeed four distinct labels and this will always be true after arbitrary rotations of the cube, so the map

$$\begin{aligned} \phi : \; & O \to S_4 \\ & R \mapsto \phi(R) = \sigma_R, \end{aligned} \tag{3}$$

where σ_R is a permutation of the four distinct labels of a single, fixed face of the cube, is well defined. We notice that 180° rotations around a midpoint of an edge give rise to 2-cycles (exchanging the two adjacent vertices). We can exchange opposite vertices with such a rotation by choosing the rotation axis parallel to the face. 90° rotations around a midpoint of a face gives rise to 4-cycles. 180° gives rise to two 2-cycles, exchanging opposite vertices. Finally rotations around a vortex give rise to the 3-cycles, keeping the vortex fixed.
To prove that this is an isomorphism, we show that φ is bijective. To show injectivity, we assume that $\sigma_R = \sigma_{R'}$, so both rotations yield the same face. This implies that the entire cube looks the same since no rotation can change the fact that opposite vertices have the same label. This implies that $R = R'$ which shows injectivity. Since $|O| = |S_4| = 24$ this implies bijectivity.