Problem 1 Symmetric group

1. Let G be a group and S_N the symmetric group, i.e., the group of permutations of $X = \{1, \ldots, N\}$. Show that any group homomorphism $\varphi : G \to S_N$ induces a group action on X by the action $g \cdot x = \varphi(g)(x)$. Here the action of an element $\sigma \in S_N$ on $x \in X$ is denoted by $\sigma(x)$.

2. Consider $\sigma \in S_4$ and the action of $G = \mathbb{Z}$ defined by the homomorphism
$$\begin{aligned}\varphi_\sigma : \mathbb{Z} &\to S_4 \\ j &\mapsto \sigma^j,\end{aligned} \tag{1}$$
where we interpret negative powers as powers of the inverse permutation. Find the orbits of this group action for the permutations σ of the following form.
$$\begin{array}{c|cccc} j & 1 & 2 & 3 & 4 \\ \hline \sigma_1(j) & 3 & 1 & 2 & 4 \end{array} \quad \begin{array}{c|cccc} j & 1 & 2 & 3 & 4 \\ \hline \sigma_2(j) & 3 & 4 & 1 & 2 \end{array} \quad \begin{array}{c|cccc} j & 1 & 2 & 3 & 4 \\ \hline \sigma_3(j) & 4 & 2 & 1 & 3 \end{array}. \tag{2}$$

3. One can denote the elements of S_N more efficiently by cycles. A cycle is defined by a $\mathbb{Z}$-orbit of a permutation with the elements written out in the order in which they occur. For example, the cycles (142)(35)(6) corresponds to the following permutation in S_6
$$1 \to 4, 4 \to 2, 2 \to 1, 3 \to 5, 5 \to 3, 6 \to 6 \tag{3}$$
Any permutation is uniquely characterized by its cycles. Cycles of length one (like (6) before) are sometimes omitted. Write out $\sigma_1, \sigma_2, \sigma_3$ and their inverses in terms of their cycles.

4. Which of the permutations σ_1, σ_2, σ_3 are in the same conjugacy class? Find $\tau \in S_4$ that relates two of the above elements by $\sigma' = \tau\sigma\tau^{-1}$ if they are in the same conjugacy class.

5. Use the intuition gained by this example to show that in general for $\sigma, \tau \in S_N$ with $\sigma = (j_1, \ldots, j_k)$ a k-cycle, $\tau\sigma\tau^{-1} = (\tau(j_1), \ldots, \tau(j_k))$.

6. The cycle type of a permutation is given by the lengths of all of the disjoint cycles that appear in its decomposition. Prove that the conjugation classes of S_N are defined by the cycle type.

7. The cycle type can be visualized by so called Young diagram: Draw each k-cycle as a row of k squares, all cycles stacked on top of each other with larger cycles on the top, aligned on the left. For S_4, identify the diagrams describing σ_1, σ_2, and σ_3 and find the diagrams for the remaining conjugacy classes in S_4. How many elements does each conjugacy class have? Notice that a Young diagram of S_N defines a partition of N (i.e. a way how to write $N = 1 + 1 + 2 + 2 + 3 + 5 + ...$) and that the number of conjugacy classes of S_N is equal to $p(N)$, the number of partitions of N.

8. (Bonus question) Find the minimal set of generators of S_N. The generators of a group are those elements that by repeated multiplication produce all other elements of the group. For example, the generators of the symmetry group of the regular triangle D_3 are rotations r by $2\pi/3$ and reflections s about any symmetry axis. One usually writes $D_3 = \langle r, s \rangle$.