Problem 1 Burnside's theorem

1. Given an action of a finite group G on a set $\mathcal{E}$, show that the number of orbits r of G in $\mathcal{E}$ corresponds to the average size of the fixed point sets $\mathrm{FP} = \{x \in \mathcal{E} | g \cdot x = x\}$:

$$r = \frac{1}{|G|} \sum_{g \in G} |\mathrm{FP}(g)|. \tag{1}$$

2. How many different (fictitious) tri-atomic molecules with regular triangular shape can you construct by choosing each atom from a set of k different atomic species?

Solution to Problem 1:

1. There are at least two ways to proceed. We describe here these two methods.
First method Count the number of elements in

$$A = \{(g, x) \in G \times \mathcal{E} \mid g.x = x\} .$$

We have $|A| = \sum_{g \in G} |\mathrm{FP}(g)|$ and on the other hand,

$$\begin{aligned} |A| &= \sum_{x \in \mathcal{E}} |\mathrm{Stab}_G(x)| \\ &= |G| \sum_{x \in \mathcal{E}} \frac{1}{|\mathrm{Orb}_G(x)|} \\ &= |G| \sum_{i=1}^{r} \sum_{y_i \in \mathrm{Orb}_G(x_i)} \frac{1}{|\mathrm{Orb}_G(y_i)|} \\ &= |G| r . \end{aligned}$$

(we used that $y_i \in \mathrm{Orb}_G(x_i)$ implies $\mathrm{Orb}_G(x_i) = \mathrm{Orb}_G(y_i)$). Equating both results yields Eq. (1).
Second method We start from the formula given in the lecture:

$$\frac{1}{|G|} \sum_{g \in G \backslash \{e\}} |\mathrm{FP}(g)| = \sum_{i=1}^{r} \left(1 - \frac{1}{|\mathrm{Stab}_G(x_i)|}\right) . \tag{2}$$

We obtain

$$r = \frac{1}{|G|} \sum_{g \in G \backslash \{e\}} |\mathrm{FP}(g)| + \sum_{i=1}^{r} \frac{1}{|\mathrm{Stab}_G(x_i)|}. \tag{3}$$

Using $|G|/|\mathrm{Stab}_G(x_i)| = |\mathrm{Orb}_G(x_i)|$, we can write

$$\sum_{i=1}^{r} \frac{1}{|\mathrm{Stab}_G(x_i)|} = \frac{1}{|G|} \sum_{i=1}^{r} |\mathrm{Orb}_G(x_i)| = \frac{|\mathcal{E}|}{|G|} = \frac{|\mathrm{FP}(e)|}{|G|}. \tag{4}$$

Inserting this back into Eq. (3) yields Eq. (1).

2. Any two configurations (i_1, i_2, i_3) describe the same molecule if they are related by a group operation from D_3 (it is the same molecule looked at from a different angle). We must avoid counting these configurations more than once. We define a group action from the group D_3 onto the set $\mathcal{E} = \{(i_1, i_2, i_3) \mid 1 \le i_1, i_2, i_3 \le k\}$. The total number of actually distinct molecules is given by the number of orbits of this group action. We use Burnside's theorem to count the orbits via the cardinality of the fixed point sets. The group consists of $\{e, r, r^2, f_1, f_2, f_3\}$ and we have $|\mathrm{FP}(e)| = |\mathcal{E}| = k^3$. Since the rotations r and r^2 move all the corners of the triangle, their fixed points are triangles of equal atoms in all corners. This yields $|\mathrm{FP}(r)| = k$. Finally, the three mirror operations f_j leave a single atom invariant (which thus can be of any species to be in the fixed point set) while exchanging the two other atoms (which thus must be of the same species). The total number of fixed points is $|\mathrm{FP}(f_j)| = k^2$. Summing over all elements, we obtain from Eq. (1)
$$r = \frac{1}{6}(k^3 + 2 \cdot k + 3 \cdot k^2). \tag{5}$$

Problem 2 Platonic solids

1. Let $G \le \mathrm{SO}_3(\mathbb{R})$ be the rotational symmetry group of some Platonic solid. Show that the number of faces times the number of edges per face is equal to the group order
$$|G| = (\#\text{faces})(\#\text{edges per face}). \tag{6}$$
Vocabulary: faces (areas, 2 dim.), edges (lines, 1 dim.), vertices (points, 0 dim.)
Hint: Use the group action of G on the set of faces together with the fact that all faces of a Platonic solid are equal polygons.
2. Express $|G|$ in terms of the number of vertices or the number of edges.

Solution to Problem 2:

1. Let F be the set of faces of the Platonic solid. Since all faces are equal, we can reach any face from any other face by means of a rotation $g \in G$. This means that the set F contains only a single G-orbit, i.e., $F = \mathrm{Orb}_G(x)$ for any $x \in F$. We have learned in the lecture (Theorem 2.8) that in this case there exists an isomorphism
$$G/\mathrm{Stab}_G(x) \simeq F, \tag{7}$$
and in particular $|G| = |F||\mathrm{Stab}_G(x)|$, where $|F| = \#\text{faces}$. To determine $|\mathrm{Stab}_G(x)|$, recall that each face is a regular n-gon, where $n = (\#\text{edges/face})$. The symmetries of an n-gon are described by the dihedral group D_n (of order $2n$), but only the n rotations are contained in G. Therefore $|\mathrm{Stab}_G(x)| = n$, which proves the statement.
2. Since also all vertices are the same, the same argument yields
$$|G| = (\#\text{vertices})(\#\text{edges/vertex}). \tag{8}$$
Finally, also all edges look the same. They always have only two stabilizers (rotations of 180° and identity), so we also get
$$|G| = 2(\#\text{edges}). \tag{9}$$

Problem 3 Reconstruction of the octahedron

Show that the finite subgroup $G \leq \mathrm{SO}_3(\mathbb{R})$ with 3 orbits when acting on the set of its fixed points on the unit sphere, defined by the fact that an element x_i in orbit i has a stabilizer with $(n_1, n_2, n_3) = (2, 3, 4)$ elements, where $n_i := |\mathrm{Stab_G(x_i)}|$, corresponds to O, the rotational symmetry group of the octahedron/cube. Follow the steps outlined below:

1. Show that $|G| = |O|$. You can use the result from the lecture notes:
$$\frac{1}{|G|} \sum_{g \in G \setminus \{e\}} |\mathrm{FP}(g)| = \sum_{i=1}^{r} \left(1 - \frac{1}{|\mathrm{Stab}_G(x_i)|}\right). \tag{10}$$
2. Pick any x_3 in the third orbit. What is the structure of the stabilizer of x_3?
3. As you proceed, depict the elements of $\mathrm{Orb}_G(x_3)$ on the unit sphere. To do that, decompose them into orbits of $H = \mathrm{Stab}_G(x_3)$. Show that they describe the vertices of an octahedron. Show that $G = O$.
4. What is the meaning of the other two orbits? How can you reconstruct the cube?

Solution to Problem 3:

1. The action of a subgroup $G \leq SO_3(\mathbb{R})$ on the sphere has exactly two fixed points for each element of G except for the identity e. Eq. (10) therefore reduces to
$$2 - \frac{2}{|G|} = \sum_{i=1}^{3} \left(1 - \frac{1}{n_i}\right) \tag{11}$$
which for $(n_1, n_2, n_3) = (2, 3, 4)$ yields $|G| = 24$. The octahedron has 8 triangular faces which according to Ex. 2 yields $|O| = 8 \times 3 = 24$. It is dual to the cube which has 6 squared faces with $6 \times 4 = 24$.
2. The stabilizer of x_3 is a group of rotations, all of which leave x_3 invariant. This means that all these rotations are about the axis $(x_3, -x_3)$. We have $\mathrm{Stab}_G(x_3) \leq G$. Since these are all rotations about the same axis, we must have a cyclic subgroup $\mathrm{Stab}_G(x_3) = \{e, h, h^2, h^3\}$, with h a $90°$ rotation.
3. First we notice that this decomposition is possible: Since $\mathrm{Orb}_G(x_3)$ is closed under the action of G, it must also be closed under the action of any $H \leq G$.
There are in total $|\mathrm{Orb}_G(x_3)| = |G|/n_3 = 6$ elements in this orbit. Clearly x_3 is its own H-orbit. If $-x_3$ is in $\mathrm{Orb}_G(x_3)$ it would also be its own H orbit. For any other point $g \cdot x_3 \in \mathrm{Orb}_G(x_3)$ that is not fixed by H (in particular $g \neq e$), we have an H-orbit with 4 elements $\mathrm{Orb}_H(g \cdot x_3) = \{g \cdot x_3, hg \cdot x_3, h^2 g \cdot x_3, h^3 g \cdot x_3\}$. To find all 6 elements, there must be another orbit, different from x_3 with only one single element, thus $-x_3 \in \mathrm{Orb}_G(x_3)$. This means that we have three H-orbits in $\mathrm{Orb}_G(x_3)$: $\{x_3\}$, $\{-x_3\}$, and $\{g \cdot x_3, hg \cdot x_3, h^2 g \cdot x_3, h^3 g \cdot x_3\}$ for some $g \in G \backslash H$.
It remains to be shown that the orbit of $g \cdot x_3$ must be on the equator (half-way between x_3 and $-x_3$).
To achieve this final element of the proof, we repeat the same line of arguments for the element $g \cdot x_3$ and decompose the orbit into orbits of its stabilizer group.

It follows that $-g \cdot x_3 \in \mathrm{Orb}_G(x_3)$, and hence it must correspond to one of the six points that we identified before. Since $-g \cdot x_3$ is different from x_3 and $-x_3$, it must be part of the orbit with four elements. This is only possible if this orbit lies on the equator and $-g \cdot x_3 = h^2 g \cdot x_3$.
Since G maps the vertices of the octahedron onto themselves, it follows that $G \leq O$. Since $|G| = |O|$, we have shown that $G = O$.

Remark : One can be puzzled by the fact that the proof above is not purely group theoretic, but involves notions of geometry. This is absolutely normal, and should be expected: the property that G has orbits of lengths (6,4,3) by itself does not chartacterize the octahedron group! It is also crucial to use at some point the assumption that G is a subgroup of SO(3). Since we don't use at all the matrix description of G in this exercise, this assumption should be used geometrically. In other words, we have to use the fact that each element of G is a rotation characterized by an axis and an angle of rotation.

4. The other orbits describe rotations the faces and the edges of the octahedron. The edges have always only two stabilizers since they will map to themselves only under rotations of 180° (and identity). The cube can be reconstructed analogously, but instead of associating the set of poles to the vertices, we associate them to the midpoints of faces.