Problem 1 Burnside's theorem

1. Given an action of a finite group G on a set $\mathcal{E}$, show that the number of orbits r of G in $\mathcal{E}$ corresponds to the average size of the fixed point sets $\mathrm{FP} = \{x \in \mathcal{E} | g \cdot x = x\}$:

$$r = \frac{1}{|G|} \sum_{g \in G} |\mathrm{FP}(g)|. \tag{1}$$

2. How many different (fictitious) tri-atomic molecules with regular triangular shape can you construct by choosing each atom from a set of k different atomic species?

Problem 2 Platonic solids

1. Let $G \leq \mathrm{SO}_3(\mathbb{R})$ be the rotational symmetry group of some Platonic solid. Show that the number of faces times the number of edges per face is equal to the group order

$$|G| = (\#\text{faces})(\#\text{edges per face}). \tag{6}$$

   Vocabulary: faces (areas, 2 dim.), edges (lines, 1 dim.), vertices (points, 0 dim.)

   Hint: Use the group action of G on the set of faces together with the fact that all faces of a Platonic solid are equal polygons.

2. Express $|G|$ in terms of the number of vertices or the number of edges.

Problem 3 Reconstruction of the octahedron

Show that the finite subgroup $G \leq \mathrm{SO}_3(\mathbb{R})$ with 3 orbits when acting on the unit sphere, defined by the fact that an element x_i in orbit i has a stabilizer with $(n_1, n_2, n_3) = (2, 3, 4)$ elements, where $n_i := |\mathrm{Stab_G(x_i)}|$, corresponds to O, the rotational symmetry group of the octahedron/cube. Follow the steps outlined below:

1. Show that $|G| = |O|$. You can use Eq. (2.11) from the lecture notes:

$$\frac{1}{|G|} \sum_{g \in G \setminus \{e\}} |\mathrm{FP}(g)| = \sum_{i=1}^{r} \left(1 - \frac{1}{|\mathrm{Stab}_G(x_i)|}\right). \tag{10}$$

2. Pick any x_3 with $|\mathrm{Stab}_G(x_3)| = 4$, i.e., belonging to the third orbit. What are the stabilizers of x_3?
3. As you proceed, depict the elements of $\mathrm{Orb}_G(x_3)$ on the unit sphere. To do that, decompose them into orbits of $H = \mathrm{Stab}_G(x_3)$. Show that they describe the vertices of an octahedron. Show that $G = O$.
4. What is the meaning of the other two orbits? How can you reconstruct the cube?