Problem 1 All finite groups of up to 5 elements

Find all finite groups up to order 5 by using results of the last TD or by constructing all valid Cayley tables of up to 5 elements (each element has to appear exactly once in each row and once in each column). Show that the groups are Abelian and identify the different groups by isomorphisms to (products of) the cyclic groups $\mathbb{Z}/n\mathbb{Z}$.

Problem 2 Union of groups

Let G_1 and G_2 be two subgroups of G. In the lecture you saw that the intersection of two subgroups always defines a group. When does the union $G_1 \cup G_2$ give rise to a group?

Problem 3 Quotient groups

1. Remember that the set of left costs G/H only forms a group, called quotient group, if $H \triangleleft G$, i.e. if H is a normal subgroup of G. Given a subgroup of this quotient group $A \leq G/H$, show that we can reduce G to a subgroup $G' \leq G$ with $H \triangleleft G'$ such that A can be also written as a quotient group $A = G'/H$.
2. Let $\varphi : G \to G'$ be a group homomorphism. Show that
$$\mathrm{Im}(\varphi) \simeq G/\mathrm{Ker}(\varphi). \tag{2}$$
3. Show that
$$G/Z(G) \simeq \mathrm{Inn(G)}, \tag{3}$$
where Inn(G) is the group of inner automorphisms of G. An inner automorphism is defined as
$$\varphi_g : G \to G \tag{4}$$
$$h \mapsto \varphi_g(h) = g \cdot h \cdot g^{-1}. \tag{5}$$

Problem 4 Harder questions about normal subgroups

1. Is the relation $\triangleleft$ (i.e. the property of being a normal subgroup) transitive? In other words, is it true that if H, F and G are groups such that $H \triangleleft F \triangleleft G$, then $H \triangleleft G$?
2. Let G be a finite group, and H be a subgroup of G of prime index p. Show that if no prime smaller than p divides $|G|$, then H is a normal subgroup.