Problem 1 Cayley tables

The composition law of finite groups $\mathcal{G} = \{g_1, \ldots, g_N\}$ can be described by Cayley tables of the form:

$$\begin{array}{c|ccccc} & g_1 & \cdots & g_i & \cdots & g_N \\ \hline g_1 & & & & & \\ \vdots & & & \vdots & & \\ g_j & & \cdots & g_j \cdot g_i & \cdots & \\ \vdots & & & \vdots & & \\ g_N & & & & & \end{array}$$

1. Show that each element of $\mathcal{G}$ appears exactly once in each row and each column in the Cayley table.
2. Prove Cayley's theorem: Each group of N elements is isomorphic to a subgroup of the symmetric group S_N (i.e., the set of all permutations of N elements).

Problem 2 The group D_3

The group D_3 describes all symmetries (rotations and mirror operations) of an equilateral triangle.

1. Construct the Cayley table of D_3. Is the group Abelian?
2. Find all subgroups of D_3. Whenever possible, construct the corresponding quotient group and its Cayley table. Find the left- and right cosets of some non-normal subgroup.

Problem 3 Lagrange's theorem

Show that for a finite group $\mathcal{G}$ with subgroup $\mathcal{H} \leq \mathcal{G}$, the order $|\mathcal{H}|$ divides $|\mathcal{G}|$.

Problem 4 Modular arithmetics

1. Construct the quotient groups of $(\mathbb{Z}, +)$.
2. Show that for any $g \in \mathcal{G}$, where $\mathcal{G}$ is a finite group, $P_g = \{k \in \mathbb{Z} | g^k = e\}$ can be written as $|g|\mathbb{Z}$. The number $|g|$ is called the order of g.
3. Show that $|g|$ divides $|\mathcal{G}|$.
4. Show that every group $\mathcal{G}$ whose order $|\mathcal{G}| = p$ is prime is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.