TD 7

The Schwarzschild Black hole

We define a manifold to be *maximal* if every geodesic emanating from en arbitrary point either can be extended to infinite values of the affine parameter or terminates on an intrinsic singularity. We recall the Schwarzschild metric, which is *a priori* valid for $r > 2m$:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) .$$

The goal of this exercise is to find the maximal extension of the space-time described by this metric, and to understand its causal structure.

1. Show that the equation of the radial null geodesics are

$$\pm t = r + 2m\log|r - 2m| + C \tag{7.1}$$

   where C is a constant and $r > 2m$. Draw the corresponding space-time diagram and make the light-cones explicit.

2. Show that (7.1) is also the equation of radial null geodesics in the region $0 < r < 2m$ and complete the space-time diagram. What is the problem ? In the following we try to understand what really happens in the region $0 < r < 2m$.

3. Consider a radially infalling particle, which we drop from infinity with zero initial velocity. Show that it satisfies

$$\begin{cases} \left(1 - \frac{2m}{r}\right)\dot{t} = 1 \\ \left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 = 1 \end{cases} \tag{7.2}$$

   where the dot is differentiation with respect to the proper time τ along the trajectory. Show that this implies

$$\tau - \tau_0 = \frac{2}{3\sqrt{2m}}(r_0^{3/2} - r^{3/2})\,. \tag{7.3}$$

   Compare with the classical result. What is the physical meaning of this equation ?

4. Show that, if we instead parametrise the motion in term of the Schwarzschild time t, we have

$$\begin{aligned} t - t_0 \;&=\; -\frac{2}{3\sqrt{2m}}\left(r^{3/2} - r_0^{3/2} + 6m(\sqrt{r} - \sqrt{r_0})\right) \\ &\quad +2m\log\frac{(\sqrt{r} + \sqrt{2m})(\sqrt{r_0} - \sqrt{2m})}{(\sqrt{r_0} + \sqrt{2m})(\sqrt{r} - \sqrt{2m})} \end{aligned} \tag{7.4}$$

5. Show (or admit) that
$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}\,. \tag{7.5}$$
What is the value of the Ricci scalar R ?

6. Is the manifold parameterized by $(t,r) \in \mathbb{R}\times]2m,\infty[$ with the Schwarzschild metric maximal?

7. We have seen that the problem of the metric at $r = 2m$ is a question of coordinates. We would like to use more adapted coordinates. Explain what is the interest of defining the so-called Eddington-Finkelstein time
$$\bar{t} = t + 2m\log(r-2m) \tag{7.6}$$
for $r > 2m$.

8. Show that the metric then reads
$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}\bar{t}^2 + \frac{4m}{r}\mathrm{d}\bar{t}\mathrm{d}r + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{7.7}$$
What happens at $r = 2m$? Draw the $(r,\bar{t})$ diagram, with ingoing and outgoing photons, light-cones as well as an the trajectory of an infalling massive particle.

9. We define $v = \bar{t} + r$. Write down the metric in coordinates (v, r, θ, ϕ).

10. Why is it also interesting to define $t^* = t - 2m\log(r-2m)$, and , $w = t^* - r$? Coordinates $\bar{t}$ and t^*, and the associated v and w, are commonly known as *Eddington-Finkelstein* coordinates.

11. Show that the metric can be written
$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}v\mathrm{d}w + r^2\mathrm{d}\Omega^2 \tag{7.8}$$
$$= \frac{16m^2}{r}\exp\left(-\frac{r}{2m}\right)(-\mathrm{d}t'^2 + \mathrm{d}r'^2) + r^2\mathrm{d}\Omega^2 \tag{7.9}$$
where we have introduced the *Kruskal* coordinates
$$v' = \exp\left(\frac{v}{4m}\right) = t' + x' \tag{7.10}$$
$$w' = -\exp\left(-\frac{w}{4m}\right) = t' - x' \tag{7.11}$$
and the variable r is defined implicitly by the equation
$$t'^2 - x'^2 = (2m - r)\exp\left(\frac{r}{2m}\right)\,. \tag{7.12}$$

12. Draw the space-time diagram in coordinates (t', x').

13. To study the causal structure of this space-time, we (again !) define new coordinates, called the *compact* Kruskal coordinates, by the relations
$$\tan v'' = \frac{v'}{\sqrt{2m}} \qquad \tan w'' = \frac{w'}{\sqrt{2m}} \tag{7.13}$$
Draw the associated space-time diagram, with coordinates (t'', x'') defined from v'' and w'' as previously, which is called the *Penrose diagram*.

14. Draw the Penrose diagram of Minkowski space-time, and compare with the Schwarzschild Penrose diagram.