TD 6

Stars

6.1 The Schwarzschild solution

We want to describe the spacetime outside a star. For simplicity we will assume that the star is static and displays spherical symmetry. Outside the star there is no matter and no energy density, so we are going to solve the Einstein equations in the vacuum.

1. Show that the metric can be written

$$\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{6.1}$$

   where the functions ν and λ will be determined later. What is $\mathrm{d}\Omega^2$?

2. Write down the Einstein equations explicitely.

3. Using two of the equations, show that $\lambda' + \nu' = 0$, and then show that $\frac{r}{2}(1 - e^{-\lambda(r)})$ is a constant, that we will call m.

4. Deduce the explicit expression of the metric.

5. How is the constant m related to the mass of the star?

6. We consider a geodesic in this spacetime, with tangent vector u^μ normalised by $u_\mu u^\mu = -\kappa$. What are the possible values of κ?

7. Show that the quantities

$$E = \left(1 - \frac{2m}{r}\right)\dot{t} \qquad \text{and} \qquad L = r^2 \sin^2\theta\dot{\phi}$$

   are conserved along the geodesic. Show that they are associated to two particular Killing vector fields.

8. Find the effective potential for a massive particle and for a light ray.

6.2 Inside a relativistic star

Now we want to know what happens inside the star. We will assume that the star is a perfect fluid characterized by an energy density ρ, a pressure p and a 4-velocity u^μ, inside a sphere of radius R. To obtain the energy-momentum tensor $T^{\mu\nu}$ we take our reference frame as the frame in which the fluid is at rest. We keep assuming that the star is static and has spherical symmetry, and write our ansatz $\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$.

1. In this frame, write down the components of $T^{\mu\nu}$ in terms of p and ρ.
2. Using the fact that the only 2-covariant tensors are $g_{\mu\nu}$ and $u_\mu u_\nu$, write down $T^{\mu\nu}$ in a covariant way.
3. Write down the Einstein equations.
4. Show that
$$e^{-\lambda(r)} = 1 - \frac{2\mathcal{M}(r)}{r}$$
where
$$\mathcal{M}(r) = 4\pi \int_0^r \rho(s) s^2 \mathrm{d}s\,, \tag{6.2}$$
and that
$$\nu'(r) = \frac{2\mathcal{M}(r) + 8\pi p(r) r^3}{r(r - 2\mathcal{M}(r))}\,. \tag{6.3}$$
5. Use the conservation of the energy-momentum tensor to obtain the Tolman-Oppenheimer-Volkov equation
$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{\mathcal{M}(r) + 4\pi p(r) r^3}{r(r - 2\mathcal{M}(r))}(p(r) + \rho(r))\,. \tag{6.4}$$
Equations (6.2), (6.3) and (6.4) are known as the equations of structure for spherical stars.
6. ***What is the equivalent of (6.4) in classical Newtonian physics?
7. Explain why the problem can not completely be solved without another assumption.
8. In this question we assume that the energy density is constant : $\rho(r) = \rho_0$.
   (a) Is this assumption realistic? Show that it is inconsistent with special relativity.
   (b) Find the function $p(r)$.
   (c) Show that this model predicts the existence of a maximal mass. Comments?
   (d) Write down the metric inside the star.
   (e) Compare the volume inside the star to the value it would take if the spacetime were flat.
   (f) ***Find and plot an embedding diagram for a 2-dimensional slice of this spacetime geometry at constant t and $\theta = \pi/2$.
9. In this question we assume $p(r) = \rho(r)$.
   (a) Why is it necessary to add the assumption that the star is contained inside a spherical shell (not made of realistic matter) at radius R, of negligible mass? What is the area of this shell?
   (b) Find a simple solution of the equations of structure in which $\rho(r)$ is of the form $\rho(r) = k/r^n$ where k is a constant to be found and n is a positive integer.
   (c) What is the mass of the star?
   (d) What pressure does the shell have to exert?

6.3 Bounds on the maximal mass of neutron stars

We would like to prove that general relativity provides a bound on the maximal mass of a neutron star, assuming a detailed knowledge of the equation of state up to the density ρ_0 of ordinary nuclei but making only general assumptions on its properties beyond. These assumptions are :

- $\rho > 0$
- $p > 0$
- $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$.

We will not derive the best bound, but we will show the existence of this bound, using the equations of structure, (6.2), (6.3) and (6.4).

1. Give an order of magnitude for ρ_0.
2. Explain why the three assumptions above are reasonable.
3. Show that the star can be divided into a *core* where the density is higher than ρ_0 and an *envelope* where it is lower. Let r_0 be the radius of the core, and $\mathcal{M}_0 = \mathcal{M}(r_0)$. We will call $\mathcal{M}_0$ the *core mass*.
4. Using the fact that the core can't be inside its own Schwarzschild radius, show that there is a maximum core mass. Express this maximum as a function of ρ_0 only.
5. We define the numerical coefficient α by

$$\alpha = \frac{(\mathcal{M}_0)_{max}}{M_\odot}\sqrt{\frac{\rho_0}{2.9 \times 10^{14}\,\mathrm{g/cm}^3}}\,.$$

   What is the value of α for $(\mathcal{M}_0)_{max}$ found above?