TD 5

Einstein equations and the Energy-momentum tensor

5.1 Energy-momentum-stress tensor

This exercise illustrates the concept of energy-momentum-stress tensor, which is in intimate relation with the concept of density of energy and momentum. The goal is to make as clear as possible the notions of *density* and *energy* in a relativistic context. For simplicity we assume that we are in Minkowski space with the usual metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

1. Explain why we don't loose much generality with this last assumption.

2. In this question we want to define the density of a scalar, namely the *number density* of particles. We consider a box with $\mathcal{N}$ particles moving along one axis with speed V. The volume of the box in its rest frame is $\mathcal{V}_*$, the rest number density is denoted n and the number density in the frame where the box is moving is denoted N^0.

   (a) Find the relation between n, N^0 and V.

   (b) Show that N^0 is the time component of a 4-vector N^μ, and express N^μ in terms of n and the 4-velocity u^μ. The spacial part of this 4-vector is denoted $\vec{N}$.

   (c) Write down the equation of conservation involving N and $\vec{N}$, and then rewrite this equation in a covariant way, using the 4-vector N^μ.

   (d) The lesson of this discussion is that *densities of scalar quantities are the time components of 4-vectors whose spatial components are the corresponding current density*. To see this geometrically, we have to realize that a 3-volume is just a 3-surface in the 4-dimensional spacetime, and hence should be oriented by a unit vector n^μ. Show that the number of particles in the 3-volume $n^\mu \delta\mathcal{V}$ is $\delta\mathcal{N} = N^\mu n_\mu \delta\mathcal{V}$.

   (e) Interpret densities and currents as fluxes.

   (f) What is the electric charge inside a given 3-volume $\delta\mathcal{V}$?

3. Now we move on to the density of a 4-vector, like the energy-momentum 4-vector p^μ.

   (a) Why do the description of the density of energy and momentum need a tensor with 2 indices $T^{\mu\nu}$? Show that the defining equation is

   $$\delta p^\mu = T^{\mu\nu} n_\nu \delta\mathcal{V}\,. \tag{5.1}$$

   (b) Using the vector $n_\mu = (1, 0, 0, 0)$, explain what are the components T^{tt} and T^{it} (for $i = x, y, z$).

(c) Using the vector $n_\mu = (0,1,0,0)$, explain what are the components T^{tx} and T^{ix} (for $i = x, y, z$).

(d) Suppose that an observer is moving in spacetime with 4-velocity u^μ_{obs}. Show that the energy density measured by the observer is[1]

$$T_{\mu\nu} u^\mu_{obs} u^\nu_{obs} \,.$$

5.2 Electromagnetic field

We consider the electromagnetic field, described as usual by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

Here the coordinate system that is being used is (x^μ) and is general ; the spacetime is also general, meaning that the curvature may be non-zero.

1. Recall what is A_μ and what is the physical interpretation of $F_{\mu\nu}$.

2. Show that $F_{\mu\nu}$ is a tensor and that it satisfies

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0 \,.$$

3. *** Show that the energy-momentum tensor of the electromagnetic field is

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma}F_{\rho\sigma}\right) .$$

4. Write down Maxwell's equations in a covariant way, with sources represented by a current J^μ, and compute $\nabla^\mu T_{\mu\nu}$. What is the interpretation of this result?

5. Show that in a spacetime with only an electromagnetic field and no matter, Einstein equations read $R_{\mu\nu} = 8\pi G T_{\mu\nu}$.

5.3 Are wormholes physical?

We remind the metric of the wormhole that we already studied :

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (b^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \,.$$

We would like to know whether a certain distribution of matter and energy can cause this geometry through Einstein's equations.

1. Compute the Einstein tensor in this geometry.

2. Is it possible to construct the wormhole with classical means?

[1] You can consider a little volume $\delta\mathcal{V}$ in the observer's rest frame and compute the energy δE measured inside.