TD 4

Curvature 2

4.1 Mathematics of Curvature

Let E be a vector space of dimension n with a metric g. We define an *algebraic curvature tensor* as any 4-covariant tensor R_{ijkl} that satisfies the three conditions :

- $R_{ijkl} = -R_{jikl}$;
- $R_{ijkl} = -R_{ijlk}$;
- $R_{[ijk]l} = R_{ijkl} + R_{jkil} + R_{kijl} = 0$.

We would like to find the dimension of the space $\mathcal{R}E$ of algebraic curvature tensors.

1. Show that $\mathcal{R}E \subset S^2(\Lambda^2 E^*)$.
2. If $n = 2$, compute the dimension of $\mathcal{R}E$.
3. In this question we set $n = 3$.
   (a) ***Show that $\dim \mathcal{R}E = 6$.
   (b) Show that the Ricci tensor has 6 independant components.
   (c) ***These results indicate that it should be possible to express the Riemann tensor in terms of the metric and the Ricci tensor only, with no derivative. Find this expression.

Now we want to find the general formula. For any tensor $S \in S^2(\Lambda^2 E^*)$ we define the tensor $\alpha(S)$ by

$$(\alpha(S))_{ijkl} = \frac{1}{3}\left(S_{ijkl} + S_{jkil} + S_{kijl}\right) .$$

4. Show that α is an endomorphism of $S^2(\Lambda^2 E^*)$, and that moreover it is a projector.
5. Deduce from the previous question the general formula for the dimension of $\mathcal{R}E$ as a function of n.

4.2 Geodesics in the hyperbolic plane

Consider the hyperbolic plane $H = \{(x, y)|y > 0\}$ with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} .$$

We would like to find the geodesics of the hyperbolic plane.

1. Write down the geodesic equation using the Lagrangian formalism.
2. Find the quantities which are conserved along the geodesics.
3. Show that the Euclidean curvature[1] of a given geodesic is given by

$$\rho = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

   where the prime denotes the derivation with respect to a parameter τ.

4. Show that this curvature is constant along a given geodesic.
5. Describe the set of geodesics.

[1]Note that there is no intrinsic meaning of curvature for a one-dimensional manifold. Here we deal with the extrinsic curvature of the geodesic seen as a curve in $\mathbf{R}^2$.