TD 3

Curvature 1

3.1 Preliminary questions

Let's consider a spacetime endowed with a metric g with Minkowski signature $(- + ...+)$ in any dimension. Let $\Gamma^{\mu}_{\nu\rho}$ be the coefficients of the Levi-Civita connection associated to the metric g.

1. Show that
$$\Gamma^{\nu}_{\mu\nu} = \partial_\mu \log \sqrt{-g} \,. \tag{3.1}$$

2. Deduce that the expression of the Laplacian of a scalar function is
$$\Delta f = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu f \right) \,. \tag{3.2}$$

3.2 Playing with a metric

In a certain spacetime geometry the metric is

$$\mathrm{d}s^2 = -(1 - Ar^2)^2 \mathrm{d}t^2 + (1 - Ar^2)^2 \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \,.$$

In this spacetime, compute the following quantities :

1. The proper distance along a radial line from the center $r = 0$ ro a coordinate radius $r = R$.

2. The area of a sphere of coordinate radius $r = R$.

3. The 3-volume of a sphere of coordinate radius $r = R$.

4. The 4-volume of a four-dimensional tube bounded by a sphere of coordinate radius R and two planes at constant t separated by a time T.

If you want to train yourself you can also compute the Christoffel symbols, the Riemann tensor, the Ricci tensor, and the scalar curvature. As a check, for the scalar curvature you should find

$$\frac{2A^2 r^2 \left(A^2 r^4 - 4Ar^2 + 7 \right)}{\left(Ar^2 - 1 \right)^4} \,.$$

3.3 Symmetries, Lie derivatives and Killing vectors

3.3.1 The algebra of Killing vector fields

Consider a spacetime with arbitrary metric $g_{\mu\nu}$ and coordinates x^μ. We say that the vector field ξ^μ is a Killing vector field if under the change $x^\mu \to x'^\mu = x^\mu + \epsilon\xi^\mu$ the distance $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is left unchanged at first order in ϵ.

1. Write down the equation that characterize Killing vector fields, using $g_{\mu\nu}$, ξ^μ and their usual derivatives with respect to x^μ. This is called the Killing equation.

2. Show that the same result can be recovered by asking that the variation of the metric due to the change of point is identical to the one induced by the coordinate change.

3. Express the Killing equation in terms of covariant derivatives.

4. Show that a given vector field ξ^μ is a Killing vector field if and only if the Lie derivative of the metric along ξ^μ vanishes.

5. Show that a Killing vector field ξ^μ satisfies

$$D_\mu D_\nu \xi_\rho = R^\sigma{}_{\mu\nu\rho}\xi_\sigma\,. \tag{3.3}$$

6. In this question we choose a set of coordinates x^μ and assume that the metric doesn't depend on the n-th coordinate x^n. Show that the metric admits a Killing vector.

7. Show that the set of Killing vector fields is a vector space.

8. We recall that given two vector fields X^μ and Y^μ, we can define a new vector field Z^μ, called the *commutator* of X^μ and Y^μ, by

$$Z^\mu = [X,Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu\,.$$

   Show that the commutator of two Killing vector fields is a Killing vector field.

3.3.2 The example of the 2-dimensional sphere

We now take the example of a 2-dimensional variety, namely the sphere.

1. Show that the metric with the usual spherical coordinates can be written

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\,.$$

2. Write down explicitly the Killing equations.

3. Show that the space of Killing vector fields is a 3-dimensional vector space with basis

$$\begin{cases} \xi_1 = \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi \\ \xi_2 = \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi \\ \xi_3 = \partial_\phi \end{cases}$$

4. What is the algebra of these Killing vectors fields?

5. Compute the Riemann tensor, the Ricci tensor and the Ricci scalar.

3.3.3 Conserved quantities

Consider a spacetime with a metric $g_{\mu\nu}$ in a given set of coordinates. We say that a vector field I^μ is conserved if $D_\mu I^\mu = 0$, and we say that a symmetric tensor $T^{\mu\nu}$ is conserved if $D_\mu T^{\mu\nu} = 0$.

1. Let $T^{\mu\nu}$ be a symmetric and conserved tensor. Prove that if there exist a Killing vector field, then one can construct a conserved vector field I^μ.

2. We now define $\hat{I}^\mu = \sqrt{-g} I^\mu$ and

$$Q = \iiint \hat{I}^0 \mathrm{d}^3 x \, .$$

   Show that $\partial_\mu \hat{I}^\mu = 0$.

3. Show that if $\hat{I}^\mu$ decreases fast enough at the boundary of spacetime, then Q does not depend on time.