TD 2

Differential geometry

2.1 Questions

1. Let A^μ and B^ν be two 4-vectors in a given set of coordinates. We define $C^{\mu\nu} = A^\mu + B^\nu$. Are the $C^{\mu\nu}$ the components of a tensor?

2. In a certain space-time geometry the metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + 2\mathrm{d}x\mathrm{d}t + \mathrm{d}y^2 + \mathrm{d}z^2\,.$$

   Show that this space-time is the standard Minkowski space-time of special relativity.

3. Show that the definition of the covariant derivative, the assumption that the metric is covariantly constant and the fact that the Christoffel symbols are symmetric $\Gamma^i_{jk} = \Gamma^i_{kj}$ imply that

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk})\,. \tag{2.1}$$

4. Write down the transformation rule for the Christoffel symbols under a change of coordinates.

2.2 The sphere S^2

We consider the sphere

$$S^2 = \{p = (x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = 1\}$$

and we define two open sets

$$U_\pm = S^2 - \{(0, 0, \pm 1)\}\,.$$

We project the set U_+ on the equatorial plane $z = 0$ by stereographic projection from the north pole $(0, 0, 1)$, and this defines a map $p \mapsto (x, y)$, where x and y are functions of $x_{1,2,3}$. Similarly, the stereographic projection from the south pole to the equatorial plane of U_- defines a map $p \mapsto (\bar{x}, \bar{y})$.

1. In this (facultative) question, we want to describe the sphere S^2 as a (differentiable) manifold.
   (a) Find the functions x, y, $\bar{x}$ and $\bar{y}$.
   (b) What is the transition map $(x, y) \mapsto (\bar{x}, \bar{y})$?
   (c) Explain why this proves that S^2 is a differentiable manifold.

2. Show that if we endow the ambient space with the usual Euclidean metric, then the induced metric on the sphere, written using the usual polar coordinates θ and ϕ is

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,. \tag{2.2}$$

3. Compute the Christoffel symbols using two different methods.

4. We want to find the geodesics of the sphere. Let us look for a geodesic with an equation of the form $\theta = \theta(\phi)$.

   (a) Write down the geodesic equation.

   (b) Show that
   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2} - 2\cot\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 - \sin\theta\cos\theta = 0 \tag{2.3}$$

   (c) Show that f defined by $f(\theta) = \cot\theta$ satisfies a second order linear differential equation, and conclude.

2.3 Embedding diagram of a Wormhole

Let's consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (b^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,,$$

where b is a constant, $b \neq 0$. It does not represent a physically realistic spacetime, as far as is known, but it is nevertheless an interesting geometry to study.

1. What it this spacetime at very large r?

2. Why can we say that it is a static and spherically symmetric spacetime?

3. Write down the metric of a 3-dimensional slice at constant time t, then write down the metric of its equatorial plane $\theta = \pi/2$. The goal of the next questions is to visualize this curved 2-dimensional surface as a surface embedded in *flat* 3-dimensional space.

4. Why is it useful to adopt cylindrical coordinates (ρ, ψ, z) in the 3-dimensional ambient space? Write down the metric of this ambient space.

5. The embedding problem boils down to finding the three functions
   $$\begin{cases} z = z(r,\phi) \\ \rho = \rho(r,\phi) \\ \psi = \psi(r,\phi) \end{cases}.$$
   Show that we can simplify this and just choose $z = z(r)$ and $\rho = \rho(r)$. What is $\psi(r,\phi)$?

6. Show that the functions $z(r)$ and $\rho(r)$ satisfy two equations that we can solve.

7. Deduce the function $\rho(z)$ and plot the surface thus obtained.

2.4 Anti de Sitter space

We consider an ambient space of dimension $d + 2$ with metric

$$\mathrm{d}\Sigma^2 = -\mathrm{d}T_1^2 - \mathrm{d}T_2^2 + \sum_{i=1}^{d} \mathrm{d}X_i^2 \sim f_{\mu\nu}\mathrm{d}Y^\mu\mathrm{d}Y^\nu. \tag{2.4}$$

with $Y^\mu = (T_1, T_2, X_1, ..., X_d)$ In this space, we define the subspace that we call AdS_{d+1} via the equation

$$T_1^2 + T_2^2 - \sum_{i=1}^{d} X_i^2 = L^2 \tag{2.5}$$

where L is a given length.

1. Write down the metric $\mathrm{d}\Omega_{d-1}^2$ of the $d-1$ dimensional sphere S^{d-1} of radius 1 in usual Euclidean space, as a function of $d-1$ angles θ_1, ... , θ_{d-1}.
2. Make a drawing of AdS_2 embedded in the ambient space.
3. Show that AdS_{d+1} can be parametrised by two variables τ, ρ and $d-1$ angles that describe S^{d-1} in such a way that the induced metric on AdS_{d+1} is

$$\mathrm{d}s^2 = L^2 \left(-\cosh^2 \rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \, \mathrm{d}\Omega_{d-1}^2 \right) . \tag{2.6}$$

4. We define $R = L \sinh \rho$ and $T = L\tau$. Compute $\mathrm{d}s^2$ using these new coordinates.
5. Now we define χ by the relation $R = L \tan \chi$. Write down $\mathrm{d}s^2$ using the coordinates τ, χ and the angular variables. Make a drawing of AdS_3 using these coordinates.
6. What are the geodesics of the ambient space of dimension $d+2$?
7. Show that the geodesics in AdS_{d+1} can be obtained by extremizing the action

$$S = \int \mathrm{d}s \left[\frac{1}{2} \frac{\mathrm{d}Y^\mu}{\mathrm{d}s} \frac{\mathrm{d}Y_\mu}{\mathrm{d}s} + \lambda (Y^\mu Y_\mu + L^2) \right] \tag{2.7}$$

with respect to Y^μ and λ.

8. Show that the null (ie lightlike) geodesics in AdS_{d+1} are straight lines in the ambient space.
9. Show that a curve in AdS_{d+1} is a geodesic if and only if when seen as a subspace of the ambient space, it is included into a plane that contains the origin $Y^\mu = 0$.
10. Show that the area and the volume of a large sphere of radius R in AdS_{d+1} scale as R^{d-1}. Is it surprising ?