TD 1

Special Relativity

1.1 The paradox of the pole

A 20 meters pole is carried so fast in the direction of its length that is appears to be only 10 meters long in the laboratory frame. The runner carries the pole through the front door of the lab, which is just above 10 meters long. Just at the instant the head of the pole reaches the closed rear door, the front door can be closed. Then the rear door opens and the runner goes through.

1. For a while, the 20 meters pole was enclosed in the 10 meters lab! Is it paradoxical?

2. From the runner point of view, what is the length of the lab?

3. Explain what happened using a space-time diagram.

1.2 Accelerated observer in Minkowski space

One considers two observers O and O'. We assume that an inertial frame is attached to O and we work in this frame. We use the usual coordinates x^μ (in the frame of O) for observer O'.

1. Recall the definition of the 4-velocity u^μ of O'. Why is it a 4-vector? What is the 4-velocity of O? Compute the 4-velocity of O', assuming that its velocity in the inertial frame of O is $\vec{v} = v\vec{e_x}$.

2. Recall the definition of the 4-acceleration a^μ of O' in the same inertial frame. Compute this 4-acceleration in the situation of question 1.

3. From now on, we assume that O' is uniformly accelerated, meaning that $a \cdot a$ is a constant which we denote g^2, and moves along the positive x-axis of the inertial frame (he starts at the origin with 0 velocity at $t = 0$ in the inertial frame, and the clocks of O and O' are synchronised at this point). Why is this a sensible definition?

4. Express a^x and a^t in terms of u^x, u^t and g.

5. Compute the world-line of O' and plot the result.

6. After O' has left, O sends a light signal towards O'. Show that if O has waited too long, the signal will never reach O'.

7. The acceleration of O' is $g = 10\,m/s^2$, and he's heading towards the center of our galaxy, starting from the Earth, where O remains. The distance is $2 \times 10^{20}\,m$. How long does the trip take in the reference frame of the Earth? How old is O' when he reaches the destination?

1.3 Doppler effect

1. What is the energy of a photon having 4-momentum k^μ, when measured by an observer with velocity u^μ in some given inertial frame?

2. Alice moves on a line at constant speed v in the inertial frame of Bob (we do not assume that Bob is on this line). She emits a signal with frequency ν_A towards Bob. What is the frequency measured by Bob?

1.4 Electromagnetism

The electric potential V and the vector potential $\vec{A}$ can be grouped to form the 4-vector $A^\mu = (V, \vec{A})$. From this we define the electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Compute explicitly the components of $F_{\mu\nu}$ in terms of the electric field $\vec{E}$ and the magnetic field $\vec{B}$.

2. How does $F_{\mu\nu}$ transform under a Lorentz boost?

3. Deduce the transformation laws for the electric and the magnetic fields under a boost (you can take $\vec{v} = (\tanh\phi)\vec{e_x}$).

The equations of motion of a particle with mass m and charge e in an electromagnetic field A^μ are coded in the action

$$S = -m \int \mathrm{d}s - e \int A$$

where $A = A_\mu \mathrm{d}x^\mu$. We would like to derive these equations of motion, in a relativistic language, and then in a classical language.

4. Rewrite the action in terms of an integral over the proper time of the particle τ involving the velocity u^μ.

5. Show that the equations of motion are

$$m \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} = e u^\mu F_{\nu\mu} \, .$$

6. Show that the 4-velocity can be written $u^\mu = \left(\gamma, \gamma\vec{V}\right)$ and explain what are γ and $\vec{V}$.

7. Deduce the non Lorentz-invariant way of writing the equations of motion. What well-known formula do we recover?

1.5 Photo-producing a pion

One reaction for producing a pion from a proton is $\gamma + p \to n + \pi^+$.

1. Using a conservation law, find the relation between the masses of the particles and their 4-momenta.

2. Deduce the minimal energy that the photon would have to carry to produce a pion this way, in the frame where the proton is at rest.

3. Is it possible to bypass this limitation by going to another reference frame?