Un peu de théorie des champs conformes

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Abstract: Notes provisoires et inachevées pour les vidéos de théorie quantique des champs. Pour l'instant les langues sont un peu mélangées.

Contents

1 Introduction

We follow mainly the books [1, 2].

We identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, where we pick the usual coordinates, $z = x + iy \in \mathbb{C}$. Nous allons adopter un vocabulaire un peu inhabituel, appelant *scalaire libre* une fonction suffisamment régulière $X : \mathbb{R}^2 \to \mathbb{R}$. Nous ne précisons pas ici ce que nous entendons par "suffisamment régulière", on reviendra peut-être plus loin sur les hypothèses nécessaires. L'action du scalaire libre est la fonctionnelle

$$S[X] = \frac{1}{4\pi\alpha'} \int \mathrm{d}x\,\mathrm{d}y\ \left[(\partial_x X)^2 + (\partial_y X)^2\right] , \tag{1.1}$$

où α' est une constante, qui rend $S[X]$ sans dimension. On adopte des unités où $\hbar = c = 1$, et on mesurera toutes les grandeurs en unités d'énergie. Ainsi on dira qu'une masse, une énergie, une fréquence, ont dimension 1, alors qu'une longueur, un temps ont dimension -1.

Les dimensions des grandeurs intervenant dans l'action ci-dessus sont $[X] = -1$, $[z] = 0$, $[\alpha'] = -2$. L'écriture ci-dessus est un raccourci, en omettant les dépendances de X, de l'expression plus complète

$$S[X] = \frac{1}{4\pi\alpha'} \int \mathrm{d}x\, \mathrm{d}y\, \left[(\partial_x X(x,y))^2 + (\partial_y X(x,y))^2 \right] . \tag{1.2}$$

On peut aussi écrire ceci en notation complexe,

$$S[X] = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z\, \partial X(z) \bar{\partial} X(z) . \tag{1.3}$$

We use the translation

$$\partial = \partial_z = \frac{\partial_x - i\partial_y}{2} , \qquad \bar{\partial} = \partial_{\bar{z}} = \frac{\partial_x + i\partial_y}{2} \tag{1.4}$$

$$\partial_x = \partial + \bar{\partial} \qquad \partial_y = i(\partial - \bar{\partial}) \tag{1.5}$$

and

$$\mathrm{d}^2 z = \mathrm{d}z \wedge \mathrm{d}\bar{z} = (\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) = -2i\mathrm{d}x \wedge \mathrm{d}y . \tag{1.6}$$

Then

$$(\partial_x X)^2 + (\partial_y X)^2 = (\partial X + \bar{\partial} X)^2 - (\partial X - \bar{\partial} X)^2 = 4\partial X \bar{\partial} X \tag{1.7}$$

2 Classical physics

In classical physics, one derives the equations of motions by extremizing the action. Here it is clear that $S[X] \geq 0$ for any X. So we have to minimize the action. We find that the action is minimal when

$$\partial_x X(x,y) = \partial_y X(x,y) = 0 \tag{2.1}$$

for all x, y. In other words, X is constant. This is the *classical equation of motion.*

We see that the constant α' here plays no role, as it just specifies the magnitude of the action. In fact, it plays no role classically, but it controls the quantum corrections. Indeed, the basic idea of quantum field theory is that the field X can explore "all" possible configurations, each configuration being weighted by an exponential factor $\exp\left(-\frac{1}{\hbar} S[X]\right)$. When $\hbar \to 0$, any variation in $S[X]$ creates a huge variation in the weight $\exp\left(-\frac{1}{\hbar} S[X]\right)$, and it is suppressed. So what really matters is the magnitude of $S[X]$ compared to $\hbar$. The classical regime is when $S[X] >> \hbar$.

Here we adopt units in which $\hbar = 1$. This means that we compare $S[X]$ to 1, and here α' plays the central role. The classical regime is when

$$\int \mathrm{d}^2 z\, \partial X(z) \bar{\partial} X(z) >> 2\pi\alpha' . \tag{2.2}$$

When $\alpha' = 0$, this is always satisfied, and we recover classical physics. When α' is non-zero, it defined a scale in the theory (in square meters for instance). This means that a variation of X of order ∂X

3 Gaussian integrals

3.1 Basic formulas

This is a continuous generalization of the standard Gaussian integral. We remind the classical formula[1]

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}\,. \tag{3.2}$$

By a simple change of variable, we obtain the generalization

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \exp\left(-\frac{1}{2}ax^2 + bx\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right)\,. \tag{3.3}$$

Note that the maximum of $-\frac{1}{2}ax^2 + bx$ for $x \in \mathbb{R}$ is precisely $\frac{b^2}{2a}$, the expression in the exponential in the final result. This is an example of a saddle point phenomenon.

Now consider a positive definite symmetric matrix A of size $n \times n$. We have

$$\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right) = \sqrt{\frac{(2\pi)^n}{\det A}} = \frac{1}{\sqrt{\det \frac{A}{2\pi}}}\,. \tag{3.4}$$

Adding a linear term, with coefficients $\mathbf{j} = (j_1, \ldots, j_n)$, in the exponential, we obtain

$$\boxed{Z(A,\mathbf{j}) := \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{j}^T\mathbf{x}\right) = \frac{\exp\left(\frac{1}{2}\mathbf{j}^T A^{-1}\mathbf{j}\right)}{\sqrt{\det \frac{A}{2\pi}}}}\,. \tag{3.5}$$

This is a fundamental expression for all of quantum field theory. Let's see how it can be used to compute the so-called *moments*. We compute the derivatives of $Z(A,\mathbf{j})$ with respect to j_a and j_b using the two expressions in the box above:

- Using the integral form, we get

$$\frac{\partial}{\partial j_a}\frac{\partial}{\partial j_b} Z(A,\mathbf{j}) = \int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{j}^T\mathbf{x}\right) x_a x_b\,. \tag{3.6}$$

- Using the "reduced" form, we get

$$\frac{\partial}{\partial j_a}\frac{\partial}{\partial j_b} Z(A,\mathbf{j}) = \frac{1}{\sqrt{\det \frac{A}{2\pi}}} \frac{\partial}{\partial j_a}\frac{\partial}{\partial j_b} \exp\left(\frac{1}{2}\mathbf{j}^T A^{-1}\mathbf{j}\right) = \frac{\exp\left(\frac{1}{2}\mathbf{j}^T A^{-1}\mathbf{j}\right)}{\sqrt{\det \frac{A}{2\pi}}}(A^{-1})_{ab}\,. \tag{3.7}$$

[1]Evaluating this integral is a standard exercise:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\frac{1}{2}x^2} = \sqrt{\int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y\, e^{-\frac{1}{2}(x^2+y^2)}} = \sqrt{\int_{0}^{+\infty} r\,\mathrm{d}r e^{-\frac{1}{2}r^2} \int_{0}^{2\pi} \mathrm{d}\theta} = \sqrt{2\pi}\,. \tag{3.1}$$

Equating these two expressions and using (3.4) we obtain

$$\frac{\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{j}^T\mathbf{x}\right) x_a x_b}{\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right)} = \exp\left(\frac{1}{2}\mathbf{j}^T A^{-1}\mathbf{j}\right)(A^{-1})_{ab} \, . \tag{3.8}$$

Finally, setting $\mathbf{j} = 0$, one gets

$$\boxed{\frac{\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right) x_a x_b}{\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right)} = (A^{-1})_{ab}} \, . \tag{3.9}$$

Example

Prenons un exemple très concret. On considère pour A la matrice 2×2 donnée par

$$A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} . \tag{3.10}$$

On calcule

$$A^{-1} = \frac{2}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{3.11}$$

On a en utilisant (3.4):

$$\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right) = \frac{4\pi}{\sqrt{3}} \, . \tag{3.12}$$

et en utilisant (3.9),

$$\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right) x_1 x_2 = \frac{4\pi}{\sqrt{3}} \times \left(-\frac{2}{3}\right) = -\frac{8\pi}{3\sqrt{3}} \, . \tag{3.13}$$

$$\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x} \, \exp\left(-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right) x_1 x_1 = \frac{4\pi}{\sqrt{3}} \times \left(+\frac{4}{3}\right) = \frac{16\pi}{3\sqrt{3}} \, . \tag{3.14}$$

3.2 Infinite dimensional integrals

We now come back to (4.1). This is the infinite-dimensional generalization of (3.9). Indeed we can write $(\partial_x X)^2 = \partial_x(X\partial_x X) - X\partial_x^2 X$, so

$$S[X] = \frac{1}{4\pi\alpha'}\int \mathrm{d}x \, \mathrm{d}y \, \left[\partial_x(X\partial_x X) + \partial_y(X\partial_y X) - X\partial_x^2 X - X\partial_y^2 X\right] \tag{3.15}$$

We now assume that

$$\int \mathrm{d}x \, \mathrm{d}y \, \left[\partial_x(X\partial_x X) + \partial_y(X\partial_y X)\right] = 0 \, . \tag{3.16}$$

This is an assumption regarding the boundary conditions. So we have

$$S[X] = -\frac{1}{4\pi\alpha'}\int \mathrm{d}x \, \mathrm{d}y \, \left[X\partial_x^2 X + X\partial_y^2 X\right] = -\frac{1}{2}\int \mathrm{d}x \, \mathrm{d}y \, X\frac{\Delta}{2\pi\alpha'}X \, , \tag{3.17}$$

where Δ is the Laplacian operator, $\Delta = \Delta_{x,y} = \partial_x^2 + \partial_y^2$. Finally, we make a last rewriting:

$$S[X] = \frac{1}{2}\int \mathrm{d}x \, \mathrm{d}y \int \mathrm{d}x' \, \mathrm{d}y' \, X(x,y)A(x,y,x',y')X(x',y') \, , \tag{3.18}$$

where

$$A(x,y,x',y') = -\frac{1}{2\pi\alpha'}\delta(x-x')\delta(y-y')\Delta_{x',y'}\,. \tag{3.19}$$

Let's take the appearance of the Laplace operator here as an opportunity to recall some basic facts about the Laplace equation $\Delta f = -\lambda f$ for f a regular function on a bounded open set $\Omega \in \mathbb{R}^2$, with boundary conditions $f = 0$ on the boundary of Ω. The solutions to this equation can be taken to be real, with eigenvalues $\lambda > 0$, and the eigenfunctions can be orthogonalized using the Gram-Schmidt process. In other words, the operator $-\Delta$ possesses all the "nice properties" of symmetric and positive definite bilinear forms. Here we will assume that this is again the case on $\mathbb{R}^2$ even though it is not bounded, assuming appropriate boundary conditions. The analogy is now complete. Let's call K the inverse of A. This means that

$$\int \mathrm{d}x'\,\mathrm{d}y'\,A(x,y,x',y')K(x',y',x'',y'') = \delta(x-x'')\delta(y-y'')\,. \tag{3.20}$$

Putting in the expression of A,

$$-\frac{1}{2\pi\alpha'}\int \mathrm{d}x'\,\mathrm{d}y'\,\delta(x-x')\delta(y-y')\Delta_{x',y'}K(x',y',x'',y'') = \delta(x-x'')\delta(y-y'')\,, \tag{3.21}$$

so by performing the integrations,

$$-\frac{1}{2\pi\alpha'}\Delta_{x,y}K(x,y,x'',y'') = \delta(x-x'')\delta(y-y'')\,. \tag{3.22}$$

Notice that K depends only on the distance between the points (x,y) and (x'',y''), by rotation and translation invariance. So let's fix $(x'',y'') = 0$ (by translation), and use $r = |(x,y)|$. The equation becomes, using the expression $\Delta K = \frac{1}{r}\partial_r\left(r\partial_r K(r)\right)$ and intergating over a circle centered at the origin and with radius r,

$$-\frac{1}{2\pi\alpha'}\int_0^{2\pi}\mathrm{d}\theta\int_0^r r'\mathrm{d}r'\,\frac{1}{r'}\partial_{r'}\left(r'\partial_{r'}K(r')\right) = -\frac{r}{\alpha'}\partial_r K(r) = 1\,. \tag{3.23}$$

This all simplifies to $\partial_r K(r) = -\frac{\alpha'}{r}$, so $K(r) = -\alpha'\ln r + C$.

4 Energy-momentum tensor and conformal anomaly

Let us summarize what we have seen up to now. We have considered the action (1.1), or equivalently (1.3), which we will use mostly from now on. We have computed the two-point function, also called the propagator,

$$\langle X(x,y)X(x',y')\rangle = \frac{\int \mathcal{D}X\, X(x,y)X(x',y')e^{-S[X]}}{\int \mathcal{D}X\, e^{-S[X]}} \tag{4.1}$$

and found

$$\langle X(x,y)X(x',y')\rangle = \langle X(z)X(w)\rangle = -\alpha' \ln|z-w| + C \tag{4.2}$$

where $z = x+iy$ et $w = x'+iy'$, and C is a constant that we could not fix. Let us write this in complex coordinates only:

$$\langle X(z)X(w)\rangle = -\frac{\alpha'}{2}\left[\ln(z-w) + \ln(\bar{z}-\bar{w})\right] + C\,. \tag{4.3}$$

The constant in (4.3) is annoying. One way to get rid of it is to differentiate with respect to z and w. This is actually a central point: the fact that the logarithm decomposes into the sum (4.3) when complex coordinates are used is crucial. In fact, one can complexify the coordinates x and y of the plane $\mathbb{R}^2$ to make it into $\mathbb{C}^2$. Then, once this is done, one can parametrize this $\mathbb{C}^2$ using another set of coordinates, namely

$$z = x+iy \qquad \bar{z} = x-iy\,. \tag{4.4}$$

These two coordinates are no longer related by complex conjugation, except when x and y are restricted to live in $\mathbb{R}^2$. To emphasize this point, we now write (4.3) in the form

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\left[\ln(z-w) + \ln(\bar{z}-\bar{w})\right] + C\,. \tag{4.5}$$

It is now clear that the sum of the two logarithms is akin to a "separation of variables", into a holomorphic and an anti-holomorphic part. As such, taking the z derivative will not only remove the additive constant, but also will kill the $\bar{z}$ dependence. We thus have

$$\langle \partial_z X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\partial_z \ln(z-w) = -\frac{\alpha'}{2(z-w)}\,. \tag{4.6}$$

Taking now the w derivative, we get

$$\langle \partial_z X(z,\bar{z})\partial_w X(w,\bar{w})\rangle = -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\,. \tag{4.7}$$

We would get a similar expression by taking derivatives with respect to $\bar{z}$ and $\bar{w}$. However, the mixed derivatives give nothing interesting:

$$\langle \partial_z X(z,\bar{z})\partial_{\bar{w}} X(w,\bar{w})\rangle = 0\,. \tag{4.8}$$

In other words, the "holomorphic part" $\partial_z X(z,\bar{z})$ and the "anti-holomorphic part" $\partial_{\bar{z}} X(z,\bar{z})$ decouple. This decoupling becomes clear when we consider the derivatives of X, so this is what we will be doing from now on: we will focus on the fields $\partial_z X(z,\bar{z})$ and $\partial_{\bar{z}} X(z,\bar{z})$.

In a way, these fields are the elementary fields of our theory. We will use the notations ∂X and $\bar{\partial} X$.

4.1 OPE

We now come to central concept of CFTs, the operator product expansion. In (4.7), we have been able to compute exactly the propagator $\langle \partial X(z) \partial X(w) \rangle$, but this is because our integral (4.1) was purely Gaussian. In general, one should expect a much more difficult computation, impossible to perform in practice. However, for most applications, what will matter is the behavior of the propagator when z and w are very close to each other. As one can see, in the limit where $z \to w$, the propagator for the free boson diverges. This divergence is what we want to isolate and characterize. This is analog to the Taylor expansion in calculus. We write the OPE as

$$\boxed{\partial X(z,\bar{z}) \partial X(w,\bar{w}) \sim -\frac{\alpha'}{2} \frac{1}{(z-w)^2}}\,. \tag{4.9}$$

The analog for the anti-holomorphic field is

$$\boxed{\bar{\partial} X(z,\bar{z}) \bar{\partial} X(w,\bar{w}) \sim -\frac{\alpha'}{2} \frac{1}{(\bar{z}-\bar{w})^2}}\,. \tag{4.10}$$

4.2 Symmetries and the Energy-Momentum Tensor

The theory we are looking at clearly has a lot of symmetries. Let us give a very quick overview. First, we introduce the notion of Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \partial_\mu X \partial^\mu X\,, \qquad \text{so that} \qquad S = \int \mathrm{d}^2 z\, \mathcal{L}\,. \tag{4.11}$$

The fact that the theory is invariant by translation implies, by Noether's theorem (not reviewed here), that there is an associated *conserved current*, called the *energy-momentum tensor*, defined by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \partial^\nu X - \delta^{\mu\nu} \mathcal{L}\,. \tag{4.12}$$

Here we find

$$T^{\mu\nu} = \frac{1}{2\pi\alpha'} \left(\partial^\mu X \partial^\nu X - \frac{1}{2} \delta^{\mu\nu} \partial_\rho X \partial^\rho X \right)\,. \tag{4.13}$$

4.3 Other Computation of the Energy-Momentum Tensor

There are several equivalent definitions of the energy-momentum tensor. One of them is that the energy-momentum tensor characterizes the change of the action under a change of the metric. This means that we should make S depend on the metric, $S[X] \to S[X, g_{\mu\nu}]$ and we compute

$$T^{\mu\nu} = -2 \frac{\delta S}{\delta g_{\mu\nu}} \,|_{g_{\mu\nu} = g^0_{\mu\nu}}\,. \tag{4.14}$$

Here δ denotes a functional derivative, and g^0 is the undeformed metric.

Let's see what this gives on our example. The action can be written as

$$S[X] = \frac{1}{4\pi\alpha'} \int \mathrm{d}x\, \mathrm{d}y \begin{pmatrix} \partial_x X \\ \partial_y X \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x X \\ \partial_y X \end{pmatrix}\,, \tag{4.15}$$

and can clearly be generalized to an arbitrary metric

$$S[X,g_{\mu\nu}] \overset{?}{=} \frac{1}{4\pi\alpha'}\int \mathrm{d}x\,\mathrm{d}y \begin{pmatrix}\partial_x X\\ \partial_y X\end{pmatrix}^T \begin{pmatrix} g^{xx}(x,y) & g^{xy}(x,y)\\ g^{yx}(x,y) & g^{yy}(x,y)\end{pmatrix}\begin{pmatrix}\partial_x X\\ \partial_y X\end{pmatrix}. \tag{4.16}$$

More compactly,

$$S[X,g_{\mu\nu}] \overset{?}{=} \frac{1}{4\pi\alpha'}\int \mathrm{d}x\,\mathrm{d}y\,\partial_\mu X\partial_\nu X\, g^{\mu\nu}. \tag{4.17}$$

However it is not clear that this is invariant under a change of coordinates in the plane! In fact it is not, as the integration measure $\mathrm{d}x\,\mathrm{d}y$ picks up a Jacobian factor under the change of coordinates. This can be remedied by introducing a term $\sqrt{\det g}$, so our action is

$$S[X,g_{\mu\nu}] = \frac{1}{4\pi\alpha'}\int \mathrm{d}x\,\mathrm{d}y\,\sqrt{\det g}\,\partial_\mu X\partial_\nu X\, g^{\mu\nu}. \tag{4.18}$$

Now let's write the variation δS under a variation $\delta g_{\mu\nu}$ of the metric. We use

$$g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu \Longrightarrow \delta g_{\mu\nu}g^{\nu\rho} + g_{\mu\nu}\delta g^{\nu\rho} = 0\,, \tag{4.19}$$

et donc après multiplication par $g^{\sigma\mu}$ on obtient

$$\delta g^{\sigma\rho} = -g^{\sigma\mu}\delta g_{\mu\nu}g^{\nu\rho}. \tag{4.20}$$

Pour le déterminant, on utilise le fait que, si les valeurs propres sont $\lambda_1,\dots,\lambda_n$, on a

$$\delta\det g = \delta(\lambda_1\cdots\lambda_n) = \lambda_1\cdots\lambda_n\sum_i \frac{\delta\lambda_i}{\lambda_i} \tag{4.21}$$

donc $\delta\det g = \det g\,\mathrm{Tr}(g^{-1}\delta g)$, ce que l'on peut écrire avec les indices $\delta\det g = \det g g^{\mu\nu}\delta g_{\mu\nu}$. Puis on prend la racine carrée et on obtient

$$\delta\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}g^{\mu\nu}\delta g_{\mu\nu}. \tag{4.22}$$

On peut maintenant enfin varier l'action:

$$\begin{aligned} S[X,g_{\mu\nu}] &= \frac{1}{4\pi\alpha'}\int \mathrm{d}x\,\mathrm{d}y\left(\frac{1}{2}\sqrt{\det g}g^{\rho\sigma}\delta g_{\rho\sigma}g^{\mu\nu} - \sqrt{\det g}g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}\right)\partial_\mu X\partial_\nu X\\ &= \frac{1}{4\pi\alpha'}\int \mathrm{d}x\,\mathrm{d}y\,\sqrt{\det g}\delta g_{\rho\sigma}\left(\frac{1}{2}g^{\rho\sigma}g^{\mu\nu} - g^{\mu\rho}g^{\sigma\nu}\right)\partial_\mu X\partial_\nu X\\ &= \frac{1}{4\pi\alpha'}\int \mathrm{d}x\,\mathrm{d}y\,\sqrt{\det g}\delta g_{\rho\sigma}\left(\frac{1}{2}g^{\rho\sigma}\partial_\mu X\partial^\mu X - \partial^\rho X\partial^\sigma X\right). \end{aligned}$$

Et finalement on trouve pour le tenseur énergie-impulsion classique:

$$T^{\rho\sigma} = \frac{1}{2\pi\alpha'}\left(\partial^\rho X\partial^\sigma X - \frac{1}{2}g^{\rho\sigma}\partial_\mu X\partial^\mu X\right). \tag{4.23}$$

We then evaluate this for the flat metric $g^{\rho\sigma} = \delta^{\rho\sigma}$, and we recover (4.13).

4.4 Energy-momentum tensor in complex coordinates

Nous avons travaillé jusqu'ici en ayant à l'esprit les coordonnées x et y pour lesquelles la métrique était diagonale, ainsi qu'elle apparaît par exemple dans la matrice de l'équation (4.15). Mais nous avons vu précédemment qu'il était pratique de passer dans les coordonnées z et $\bar{z}$. Dans ces coordonnées, on peut calculer que $g_{zz} = g_{\bar{z}\bar{z}} = 0$ alors que $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. En conséquence, $g^{z\bar{z}} = g^{\bar{z}z} = 2$. On trouve donc (en abaissant les indices partout dans (4.23)):

$$T_{zz} = \frac{1}{2\pi\alpha'}\partial X \partial X \tag{4.24}$$

$$T_{\bar{z}z} = 0 \tag{4.25}$$

$$T_{z\bar{z}} = 0 \tag{4.26}$$

$$T_{\bar{z}\bar{z}} = \frac{1}{2\pi\alpha'}\bar{\partial} X \bar{\partial} X \tag{4.27}$$

Nous allons ici encore nous concentrer sur la partie holomorphe, et donc supprimer les indices à T_{zz}. Ce faisant, nous allons aussi nous séparer d'un facteur -2π. Ces histoires de normalisation, qui peuvent passer pour du pinaillage à ce point de l'histoire, se révèleront cruciales par la suite. Définissons donc:

$$T = -2\pi T_{zz} \qquad \bar{T} = -2\pi T_{\bar{z}\bar{z}}\,. \tag{4.28}$$

Dans notre exemple, nous avons donc

$$T = -\frac{1}{\alpha'}\partial X \partial X \qquad \bar{T} = -\frac{1}{\alpha'}\bar{\partial} X \bar{\partial} X\,. \tag{4.29}$$

4.5 Classical equations and symmetries

The classical equations of motion are obtained from the Euler-Lagrange equations applied to the Lagrangian. These equations take the form

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) = \frac{\partial \mathcal{L}}{\partial X}\,. \tag{4.30}$$

Here with (4.11), we find simply

$$\partial_\mu \partial^\mu X = 0\,. \tag{4.31}$$

In complex coordinates, this becomes (using the coefficients of the metric given above):

$$\partial\bar{\partial} X(z,\bar{z}) = 0\,. \tag{4.32}$$

The trace of the energy-momentum tensor is

$$T^\mu_\mu = g^{\mu\nu}T_{\mu\nu} = 2(T_{\bar{z}z} + T_{z\bar{z}}) = 0\,. \tag{4.33}$$

So we find that the energy-momentum tensor is traceless. This is related to the invariance of the action under conformal transformations (we have not defined them precisely yet, so let's say that it corresponds to invariance under change of scale).

4.6 The quantum problem of coincident fields

Nous sommes là face à une expression, T, qui est le carré d'un champ ∂X. Autrement dit, on a deux fois le même champ, évalué au même point:

$$\partial X(z,\bar{z})\partial X(z,\bar{z})\,. \tag{4.34}$$

Tant que cela reste classique, cela ne pose pas de problème, mais lorsqu'on considère le champ quantique, il en va autrement. Ce genre d'expression pose problème: en effet, on voit que le corrélateur de $\partial X(z,\bar{z})\partial X(z,\bar{z})$ diverge, d'après (4.9). Nous allons donc introduire une prescription qui peut sembler ad hoc au premier abord, et que nous appelons l'*ordre normal*. Pour un champ seul, l'ordre normal est inoffensif, mais pour un champ composite, il soustrait un commutateur:

$$:\partial X(z,\bar{z}):=\partial X(z,\bar{z}) \tag{4.35}$$

$$:\partial X(z,\bar{z})\partial X(w,\bar{w}):=\partial X(z,\bar{z})\partial X(w,\bar{w})-\langle\partial X(z,\bar{z})\partial X(w,\bar{w})\rangle\,. \tag{4.36}$$

Ceci est toujours valable, bien entendu, pour $z\neq w$. Maintenant, prenons la valeur moyenne de l'expression ci-dessus. On obtient clairement $\langle :\partial X(z,\bar{z})\partial X(w,\bar{w}): \rangle=0$. Autrement dit, dans un corrélateur, $:\partial X(z,\bar{z})\partial X(w,\bar{w}):$ reste bien défini lorsque w tend vers z. On s'autorisera donc, cavalièrement, à prendre cette limite, et donc à parler de $:\partial X(z,\bar{z})\partial X(z,\bar{z}):$. Ceci est une "régularisation" de l'expression singulière $\partial X(z,\bar{z})\partial X(z,\bar{z})$. Nous verrons ultérieurement ce que cela signifie plus précisément, et plus physiquement. Mais pour le moment, voyons ce que l'on peut faire de cette définition. On pose donc, par définition,

$$T(z,\bar{z})=-\frac{1}{\alpha'}\;:\partial X(z,\bar{z})\partial X(z,\bar{z}):\;. \tag{4.37}$$

Voyons maintenant ce qu'il se passe quand on place trois champs ∂X à des points distincts. Pour simplifier les notations, nous n'écrivons plus que les coordonnées holomorphes, et nous nous intéressont uniquement aux termes singuliers. On a:

$$\partial X(z)\partial X(v)\partial X(w)\sim-\frac{\alpha'}{2}\left(\frac{\partial X(w)}{(z-v)^2}+\frac{\partial X(v)}{(z-w)^2}+\frac{\partial X(z)}{(v-w)^2}\right) \tag{4.38}$$

et comme attendu, on voit des divergences dès que deux des trois champs s'approchent l'un de l'autre. Introduisons maintenant l'ordre normal:

$$:\partial X(z)\partial X(v):\;\partial X(w)\sim-\frac{\alpha'}{2}\left(\frac{\partial X(v)}{(z-w)^2}+\frac{\partial X(z)}{(v-w)^2}\right)\,. \tag{4.39}$$

On le voit, l'ordre normal a fait son travail, et il n'y a maintenant plus de divergence quand $v\to z$. On peut donc prendre cette limite:

$$:\partial X(z)\partial X(z):\;\partial X(w)\sim-\frac{\alpha'}{2}\left(\frac{\partial X(z)}{(z-w)^2}+\frac{\partial X(z)}{(z-w)^2}\right)\,. \tag{4.40}$$

En simplifiant et en rappelant la définition de $T(z)$, on a donc calculé

$$T(z)\partial X(w)\sim\frac{\partial X(z)}{(z-w)^2}\,. \tag{4.41}$$

On voit que tous les facteurs avaient été judicieusement choisis pour arriver à cette équation dans toute sa simplicité! On voudrait cependant que $X(w)$ apparaisse à droite, pour que l'équation soit de la forme $T(z)\partial X(w) = \ldots X(w)$. Pour cela, rien de plus simple, on développe en série et on ne garde que les termes qui seront singuliers, c'est-à-dire jusqu'à l'ordre 2 non inclus, étant donné le dénominateur $(z-w)^2$. On écrit donc $\partial X(z) = \partial X(w) + (z-w)\partial\partial X(w)$ et donc

$$\boxed{T(z)\partial X(w) \sim \frac{h\,\partial X(w)}{(z-w)^2} + \frac{\partial\partial X(w)}{z-w}} \text{ avec } h = 1\,. \tag{4.42}$$

We have introduced a constant $h = 1$ for the following reason. When e field ϕ satisfies the relation

$$T(z)\phi(w) \sim \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} \tag{4.43}$$

we say that the field is a *primary field of dimension h*. We will see the precise definitions later,[2] but one can already notice that this corresponds to ∂X having one partial derivative.

Similarly, one could expect that the energy-momentum tensor have dimension 2, as it has two partial derivatives in it. This will turn out to be true, but the field is nevertheless not a primary field, as the TT OPE will not have the form 4.43. There is actually a more severe divergence, and it is our main goal to compute it.

4.7 Normal order

We now look closely at the normal order prescription that we gave. It can be summarized by the following equation

$$:\mathcal{F}: = \exp\left(\frac{\alpha'}{4}\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \ln|z_{12}|^2 \frac{\delta}{\delta X(z_1,\bar{z}_1)}\frac{\delta}{\delta X(z_2,\bar{z}_2)}\right)\mathcal{F}\,. \tag{4.45}$$

We have introduced the notation $z_{12} = z_1 - z_2$. It satisfies

$$\langle :\mathcal{F}: \rangle = 0 \tag{4.46}$$

for any expression $\mathcal{F}$. Let's check what this gives for the examples we have already computed:

[2] We will say that the field ϕ is conformal primary of conformal dimensions $(h,\bar{h})$ if under any conformal map $z \mapsto w(z)$, the field transforms as

$$\phi'(w,\bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z})\,. \tag{4.44}$$

Again, we have not introduced all the necessary tools to understand this equation now, so we postpone that discussion.

- Two fields.

$$
\begin{aligned}
:\partial X(z,\bar{z})\partial X(w,\bar{w}): \;&= \left(1+\frac{\alpha'}{4}\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \ln|z_{12}|^2 \frac{\delta}{\delta X(z_1,\bar{z}_1)}\frac{\delta}{\delta X(z_2,\bar{z}_2)}\right)\partial X(z,\bar{z})\partial X(w,\bar{w}) \\
&= \partial X(z,\bar{z})\partial X(w,\bar{w}) \\
&\quad +\frac{\alpha'}{4}\partial_z\partial_w\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \ln|z_{12}|^2\, \delta(z_1-z)\delta(\bar{z}_1-\bar{z})\delta(z_2-w)\delta(\bar{z}_2-\bar{w}) \\
&\quad +\frac{\alpha'}{4}\partial_z\partial_w\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \ln|z_{12}|^2\, \delta(z_1-w)\delta(\bar{z}_1-\bar{w})\delta(z_2-z)\delta(\bar{z}_2-\bar{z}) \\
&= \partial X(z,\bar{z})\partial X(w,\bar{w}) + \frac{\alpha'}{2}\,\partial_z\partial_w \ln|z-w|^2 \\
&= \partial X(z,\bar{z})\partial X(w,\bar{w}) + \frac{\alpha'}{2}\,\frac{1}{(z-w)^2}\,. \qquad (4.47)
\end{aligned}
$$

  This is exactly the definition of the normal order that we gave above.

- Three fields. Similarly, we develop at order one the exponential.

- Four fields. This time, we need to develop at order 2. For conciseness we drop the anti-holomorphic dependence. We find

$$
\begin{aligned}
&:\partial X(z_1)\partial X(z_2)\partial X(z_3)\partial X(z_4): \;= \partial X(z_1)\partial X(z_2)\partial X(z_3)\partial X(z_4) \\
&\quad +\frac{\alpha'}{4}\times 2\times\left(\frac{\partial X(z_1)\partial X(z_2)}{z_{34}^2}+\frac{\partial X(z_1)\partial X(z_3)}{z_{24}^2}+\frac{\partial X(z_1)\partial X(z_4)}{z_{23}^2}\right. \\
&\quad \left.+\frac{\partial X(z_2)\partial X(z_3)}{z_{14}^2}+\frac{\partial X(z_2)\partial X(z_4)}{z_{13}^2}+\frac{\partial X(z_3)\partial X(z_4)}{z_{12}^2}\right) \\
&\quad +\frac{1}{2!}\left(\frac{\alpha'}{4}\right)^2\times 8\times\left(\frac{1}{z_{12}^2 z_{34}^2}+\frac{1}{z_{13}^2 z_{24}^2}+\frac{1}{z_{14}^2 z_{23}^2}\right)\,.
\end{aligned}
$$

  Note that there is a $\frac{1}{2!}$ from the expansion of the exponential, and a factor 8 which counts the various ways of taking the contractions.[3]

The last equation can be rewritten as

$$
\begin{aligned}
&\partial X(z_1)\partial X(z_2)\partial X(z_3)\partial X(z_4) \;=\; :\partial X(z_1)\partial X(z_2)\partial X(z_3)\partial X(z_4): \\
&\quad -\frac{\alpha'}{2}\left(\frac{\partial X(z_1)\partial X(z_2)}{z_{34}^2}+\frac{\partial X(z_1)\partial X(z_3)}{z_{24}^2}+\frac{\partial X(z_1)\partial X(z_4)}{z_{23}^2}\right. \\
&\quad \left.+\frac{\partial X(z_2)\partial X(z_3)}{z_{14}^2}+\frac{\partial X(z_2)\partial X(z_4)}{z_{13}^2}+\frac{\partial X(z_3)\partial X(z_4)}{z_{12}^2}\right) \\
&\quad -\frac{\alpha'^2}{4}\left(\frac{1}{z_{12}^2 z_{34}^2}+\frac{1}{z_{13}^2 z_{24}^2}+\frac{1}{z_{14}^2 z_{23}^2}\right)
\end{aligned}
$$

What would happen if we wanted to modify the right hand side from $\partial X(z_1)\partial X(z_2)\partial X(z_3)\partial X(z_4)$ to $:\partial X(z_1)\partial X(z_2):\,:\partial X(z_3)\partial X(z_4):$? Then in the computation one should ignore all the

[3] In total, there are 4! permutations of the four fields, and these are divided into three classes of unequivalent results, the three terms which appear in the bracket. So the multiplicity factor is $\frac{4!}{3} = 8$.

contractions between z_1 and z_2, and the contractions between z_3 and z_4. In other words, we get

$$\begin{aligned}
:\partial X(z_1)\partial X(z_2): :\partial X(z_3)\partial X(z_4): &= :\partial X(z_1)\partial X(z_2)\partial X(z_3)\partial X(z_4): \\
&-\frac{\alpha'}{2}\left(\frac{\partial X(z_1)\partial X(z_3)}{z_{24}^2}+\frac{\partial X(z_1)\partial X(z_4)}{z_{23}^2}+\frac{\partial X(z_2)\partial X(z_3)}{z_{14}^2}+\frac{\partial X(z_2)\partial X(z_4)}{z_{13}^2}\right) \\
&-\frac{\alpha'^2}{4}\left(\frac{1}{z_{13}^2 z_{24}^2}+\frac{1}{z_{14}^2 z_{23}^2}\right) \qquad (4.48)
\end{aligned}$$

Again, the normal order did its job: the right hand side is now regular in the limit where $z_1 \to z_2$, and $z_3 \to z_4$. Let us take this limit and set $z_1 = z_2 = z$ and $z_3 = z_4 = w$:

$$\begin{aligned}
:\partial X(z)\partial X(z): :\partial X(w)\partial X(w): &= :\partial X(z)\partial X(z)\partial X(w)\partial X(w): \\
&-2\alpha'\frac{\partial X(z)\partial X(w)}{(z-w)^2}-\frac{\alpha'^2}{2}\frac{1}{(z-w)^4} \qquad (4.49)
\end{aligned}$$

Now we use the fact that

$$\partial X(z)\partial X(w) = :\partial X(z)\partial X(w): -\frac{\alpha'}{2}\frac{1}{(z-w)^2} \qquad (4.50)$$

to get

$$\begin{aligned}
:\partial X(z)\partial X(z): :\partial X(w)\partial X(w): &= :\partial X(z)\partial X(z)\partial X(w)\partial X(w): \\
&-2\alpha'\frac{:\partial X(z)\partial X(w):}{(z-w)^2}+\frac{\alpha'^2}{2}\frac{1}{(z-w)^4} \qquad (4.51)
\end{aligned}$$

Finally, we Taylor expand in the normal order:

$$\begin{aligned}
:\partial X(z)\partial X(z): :\partial X(w)\partial X(w): &= :\partial X(z)\partial X(z)\partial X(w)\partial X(w): \qquad (4.52)\\
&-2\alpha'\frac{:\partial X(w)\partial X(w):}{(z-w)^2}-2\alpha'\frac{:\partial^2 X(w)\partial X(w):}{(z-w)}+\frac{\alpha'^2}{2}\frac{1}{(z-w)^4}
\end{aligned}$$

Now we can write the OPE, keeping only the singular terms. This removes the first normal order, which is regular by definition:

$$\begin{aligned}
:\partial X(z)\partial X(z): :\partial X(w)\partial X(w): \sim \frac{\alpha'^2}{2}\frac{1}{(z-w)^4}-2\alpha'\frac{:\partial X(w)\partial X(w):}{(z-w)^2} \\
-2\alpha'\frac{:\partial^2 X(w)\partial X(w):}{(z-w)}\,. \qquad (4.53)
\end{aligned}$$

4.8 TT OPE

The tedious computation of the previous section gives directly the OPE for the energy-momentum tensor. Divide (4.53) by α'^2, and notice that

$$\partial T(w) = -\frac{2}{\alpha'}\ :\partial^2 X(w)\partial X(w): \qquad (4.54)$$

to get

$$\boxed{T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}} \tag{4.55}$$

where we have introduced

$$\boxed{c = 1}\,. \tag{4.56}$$

This constant c is called the *central charge*, it is of uttermost importance for us. We will see later that the name of central charge comes from algebraic considerations. But we should mention now that it has a very strong and important physical meaning. It is our first encounter with an *anomaly*, and we devote the next subsection to some aspects of this difficult notion.

4.9 Conformal Anomaly

Anomalies are one of the most fundamental features of quantum field theories, they are subtle and delicate, yet can kill theories in certain cases, and they often have a plethora of interpretations. They also appear in mathematics in extreme circumstances, and require particular care to handle them. This section aims at giving a flavor of what anomalies in general are, by focusing on the example of the conformal anomaly c found above. We don't provide proofs of the statements, the reader is referred to [1] for instance.

We give several consequences of the conformal anomaly:

- First, the field T is not a conformal primary, as mentioned before. So under a conformal transformation, it will not obey the rule (4.44), but a modified transformation rule, which involves the so-called *Schwarzian derivative*.

- "The appearance of the central charge c, also known as the conformal anomaly, is related to a soft breaking of the conformal symmetry by the introduction of a macroscopic scale into the system. In other words, c describes the way a specific system reacts to macroscopic length scales introduced, for instance, by boundary conditions." ([1], section 5.4.2). For instance, putting the theory on a cylinder of radius R by a conformal mapping from the plane induces a change in the energy-momentum tensor, which changes the value of the vacuum energy. More precisely, we have

$$\begin{pmatrix}\text{Vacuum energy density}\\ \text{on the cylinder}\end{pmatrix} - \begin{pmatrix}\text{Vacuum energy density}\\ \text{on the plane}\end{pmatrix} = -\frac{c}{24R^2}\,. \tag{4.57}$$

  This is known as a *Casimir energy*, which is a change in vacuum energy density induced by the boundary conditions.

- The Casimir energy is related to another appearance of the anomaly, this time in mathematics, in the "summation" of divergent series. For instance, equation (4.57) is related to the regulated sum

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12}\,. \tag{4.58}$$

We will see this more clearly in the next section when we introduce the operator formalism.

- Another way of introducing a scale in the system is to curve the space, by changing the metric. In that case, the tracelessness of the energy-momentum tensor (4.33) ceases to hold, indicating indeed that scale invariance is broken. Indeed we find

$$\langle T^\mu_\mu \rangle = \frac{c}{24\pi}\mathbf{R} \tag{4.59}$$

where now $\mathbf{R}$ is the curvature (the Riemann tensor, but in 2d it is just a number).
where now $\mathbf{R}$ is the curvature (the Riemann tensor, but in 2d it is just a number).

5 Operator formalism

The normal order can be defined as

$$:a(z)b(z):= \oint \frac{\mathrm{d}w}{2\pi i}\,\frac{a(w)b(z)}{w-z}\,. \tag{5.1}$$

Operators:

$$A = \oint a(z)\mathrm{d}z \qquad B = \oint a(w)\mathrm{d}w \tag{5.2}$$

OPE and commutators:

$$[A, b(w)] = \oint_w \mathrm{d}z\, a(z)b(w)\,. \tag{5.3}$$

$$[A, B] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z\, a(z)b(w)\,. \tag{5.4}$$

5.1 Mode expansion

Scalar field modes

$$X(z,\bar{z}) = x - i\frac{\alpha'}{2}p\ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}-\{0\}}\frac{1}{m}\left(\frac{\alpha_m}{z^m}+\frac{\tilde{\alpha}_m}{\bar{z}^m}\right) \tag{5.5}$$

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{\alpha_m}{z^{m+1}} \qquad \alpha_0 = \sqrt{\frac{\alpha'}{2}}p \tag{5.6}$$

$$\bar{\partial} X(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{m\in\mathbb{Z}}\frac{\tilde{\alpha}_m}{\bar{z}^{m+1}} \qquad \tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}p \tag{5.7}$$

Commutation relations:

$$[\alpha_m, \alpha_n] = m\,\delta_{m+n} \tag{5.8}$$
$$[\tilde{\alpha}_m, \tilde{\alpha}_n] = m\,\delta_{m+n} \tag{5.9}$$
$$[\alpha_m, \tilde{\alpha}_n] = 0 \tag{5.10}$$
$$[x, p] = i\,. \tag{5.11}$$

Energy-momentum tensor modes

$$T(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} \qquad \bar{T}(\bar{z}) = \sum_{m\in\mathbb{Z}} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \tag{5.12}$$

$$L_m = \oint \frac{\mathrm{d}z}{2\pi i} z^{m+1} T(z) \tag{5.13}$$

In the case of the free scalar field we find

$$L_m = \begin{cases} \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\alpha_n & m \neq 0 \\ \frac{\alpha' p^2}{4} + \sum_{n\geq 1} \alpha_{-n}\alpha_n & m = 0 \end{cases} \tag{5.14}$$

5.2 Virasoro algebra

We now compute the commutator of the modes L_m.

$$\begin{aligned}
[L_m, L_n] &= \left[\oint \frac{\mathrm{d}z}{2\pi i} z^{m+1} T(z), \oint \frac{\mathrm{d}w}{2\pi i} w^{n+1} T(w)\right] \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} \oint_w \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+1} \oint_w \frac{\mathrm{d}z}{2\pi i} z^{m+1} \left(\frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+1} \oint_0 \frac{\mathrm{d}z}{2\pi i} (z+w)^{m+1} \left(\frac{c}{2z^4} + \frac{2T(w)}{z^2} + \frac{\partial T(w)}{z}\right) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+1} \left(\frac{c}{2}\binom{m+1}{3} w^{m-2} + 2T(w)\binom{m+1}{1} w^m + \partial T(w)\binom{m+1}{0} w^{m+1}\right) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i} \left(\frac{cm(m^2-1)}{12} w^{n+m-1} + 2(m+1)T(w) w^{n+m+1} + \partial T(w) w^{n+m+2}\right) \\
&= \frac{cm(m^2-1)}{12}\delta_{m+n} + 2(m+1)L_{n+m} - \oint_0 \frac{\mathrm{d}w}{2\pi i} T(w)(n+m+2) w^{n+m+1} \\
&= \frac{cm(m^2-1)}{12}\delta_{m+n} + 2(m+1)L_{n+m} - (n+m+2)L_{n+m} \\
&= \frac{c}{12} m(m^2-1)\delta_{m+n} + (m-n)L_{n+m}
\end{aligned}$$

We have used

$$\oint_0 \frac{\mathrm{d}z}{2\pi i} f(z) \frac{1}{z^k} = \oint_0 \frac{\mathrm{d}z}{2\pi i} \sum_n \frac{z^{n-k}}{n!} f^{(n)}(0) = \frac{1}{(k-1)!} f^{(k-1)}(0) \tag{5.15}$$

so that

$$\oint_0 \frac{\mathrm{d}z}{2\pi i} (z+w)^{m+1} \frac{1}{z^k} = \binom{m+1}{k-1} w^{m-k+2} \tag{5.16}$$

The final result is

$$\boxed{[L_m, L_n] = \frac{c}{12} m(m^2-1)\delta_{m+n} + (m-n)L_{n+m}} \tag{5.17}$$

This is the Virasoro algebra.

In particular, for $m, n \in \{1, 0, -1\}$ we find the algebra

$$[L_0, L_1] = -L_1 \tag{5.18}$$

$$[L_0, L_{-1}] = L_{-1} \tag{5.19}$$

$$[L_1, L_{-1}] = 2L_0 \,. \tag{5.20}$$

5.3 Zeta regularization

When $t \to 0$,

$$\sum_{n\geq 1}(n-\theta)e^{(n-\theta)t} = \frac{e^{t\theta}(e^t+\theta-e^t\theta)}{(1-e^t)^2} = \frac{1}{t^2} - \frac{6\theta^2-6\theta+1}{12} + O(t)\,. \tag{5.21}$$

We pick the regularization which discards the pole $\frac{1}{t^2}$ and set

$$\sum_{n\geq 1}(n-\theta) = -\frac{6\theta^2-6\theta+1}{12}\,. \tag{5.22}$$

In particular, we get

$$\sum_{n\geq 1} n = -\frac{1}{12} \qquad\qquad \sum_{n\geq 1}\left(n-\frac{1}{2}\right) = \frac{1}{24} \tag{5.23}$$

References

[1] P. Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory.* Springer Science & Business Media, 2012.

[2] J. Polchinski, *String theory: Volume 1, an introduction to the bosonic string.* Cambridge university press, 1998.